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*Mathematics*

# ON LOCALLY-BALANCED 2-PARTITIONS OF SOME CLASSES OF GRAPHS

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In this paper we obtain some conditions for the existence of locally-balanced 2-partitions with an open (with a closed) neighborhood of some classes of graphs. In particular, we give necessary conditions for the existence of locally-balanced 2-partitions of even and odd graphs. We also obtain some results on the existence of locally-balanced 2-partitions of rook's graphs and powers of cycles. In particular, we prove that if $m,n \geq 2$, then the graph $K_m \Box K_n$ has a locally-balanced 2-partition with a closed neighborhood if and only if $m$ and $n$ are even. Moreover, all our proofs are constructive and provide polynomial time algorithms for constructing the required 2-partitions.

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**Introduction.** Throughout this paper all graphs are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. The set of neighbors of a vertex $v$ in $G$ is denoted by $N_G(v)$. Let $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the maximum degree of vertices in $G$ by $\Delta(G)$. A graph $G$ is *even* (*odd*) if the degree of every vertex of $G$ is even (odd). We use the standard notations $C_n$ and $K_n$ for the simple cycle and the complete graph of $n$ vertices, respectively. A graph is a *power of cycle*, denoted $C_n^k$, if $V\left(C_n^k\right) = \{v_0, \dots, v_{n-1}\}$ and $E\left(C_n^k\right) = E_1 \cup \dots \cup E_k$, where $E_i = \{v_j v_{(j+i) \pmod n} : 0 \leq j \leq n-1\}$. Clearly, $C_n^k$ is a $2k$-regular graph.

Next we define Cartesian products of graphs. Let $G$ and $H$ be graphs. The Cartesian product $G \Box H$ of graphs $G$ and $H$ is defined as follows:

$$V(G \Box H) = V(G) \times V(H),$$

$$E(G \Box H) = \{(u_1,v_1)(u_2,v_2) \colon \ (u_1 = u_2 \wedge v_1v_2 \in E(H)) \vee (v_1 = v_2 \wedge u_1u_2 \in E(G))\}.$$

The Cartesian product $K_m \Box K_n$ is called a *rook's graph*. The terms and concepts that we do not define can be found in [1,2].

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A 2-partition of a graph $G$ is a function $f : V(G) \to \{\mathbf{0},\mathbf{1}\}$. A 2-partition $f$ of a graph $G$ is locally-balanced with an open neighborhood if for every $v \in V(G)$,

$$||\{u \in N_G(v) \colon f(u) = \mathbf{0}\}| - |\{u \in N_G(v) \colon f(u) = \mathbf{1}\}|| \leq 1,$$

where $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. A 2-partition $f'$ of a graph $G$ is locally-balanced with a closed neighborhood if for every $v \in V(G)$,

$$\big||\{u \in N_G[v] \colon f'(u) = \mathbf{0}\}| - |\{u \in N_G[v] \colon f'(u) = \mathbf{1}\}|\big| \leq 1,$$

where $N_G[v] = N_G(v) \cup \{v\}$.

We introduce some terminology and notation.

If $\varphi$ is a 2-partition of a graph $G$ and $v \in V(G)$, then define $\#(v)$, $\#[v]$ and $\varphi^*(v)$ as follows:

$$\#(v) = |\{u \in N_G(v) \colon \varphi(u) = \mathbf{0}\}| - |\{u \in N_G(v) \colon \varphi(u) = \mathbf{1}\}|,$$
$$\#[v] = |\{u \in N_G[v] \colon \varphi(u) = \mathbf{0}\}| - |\{u \in N_G[v] \colon \varphi(u) = \mathbf{1}\}|,$$

$$\varphi^*(v) = \begin{cases} -1, & \text{if } \varphi(v) = \mathbf{0}, \\ 1, & \text{if } \varphi(v) = \mathbf{1}. \end{cases}$$

Clearly, $\varphi$ is a locally-balanced 2-partition with an open neighborhood (with a closed neighborhood) if for every $v \in V(G)$, $|\#(v)| \leq 1$ ($|\#[v]| \leq 1$).

The concept of locally-balanced 2-partition of graphs was introduced by Balikyan and Kamalian [3]. Locally-balanced 2-partitions of graphs can be considered as a special case of equitable colorings of hypergraphs [4]. Berge [4] obtained some sufficient conditions for the existence of equitable colorings of hypergraphs. In [5–8], it was considered the problems of the existence and construction of proper vertex-coloring of a graph for which the number of vertices in any two color classes differ by at most one. In [9], 2-vertex-colorings of graphs, were considered for which each vertex is adjacent to the same number of vertices of every color. In particular, Kratochvil [9] proved that the problem of the existence of such a coloring is $NP$-complete even for the $(2p,2q)$-biregular ($p,q \geq 2$) bipartite graphs, i.e. bipartite graphs where all vertices in one part have degree $2p$ and all vertices in the other part have degree $2q$. In [3], Balikyan and Kamalian proved that the problem of existence of locally-balanced 2-partition with an open neighborhood of bipartite graphs with maximum degree 3 is $NP$-complete. In 2006, the similar result for locally-balanced 2-partitions with a closed neighborhood was also proved in [10]. In [11,12], the necessary and sufficient conditions for the existence of locally-balanced 2-partitions of trees were obtained. In [13], the authors obtained the necessary and sufficient conditions for the existence of locally-balanced 2-partitions of complete multipartite graphs. Recently, Gharibyan and Petrosyan [14] considered locally-balanced 2-partitions of grid-like graphs. In particular, they proved that for any $n \in \mathbb{N}$, the $n$-dimensional cube $Q_n$ has locally-balanced 2-partitions, and the torus $C_m \square C_n$ ($m,n \geq 3$) has a locally-balanced 2-partition with an open neighborhood if and only if $m \cdot n$ is even.

**Main Results.** We begin our considerations of locally-balanced 2-partitions with even and odd graphs.

**Theorem 1.** *Let G be an even graph with n vertices and*

$$k = \min\{q : v \in V(G), d_G(v) = p2^q, \text{where } p \text{ is odd and } q \in \mathbb{N}\}.$$

*If G has a locally-balanced 2-partition with an open neighborhood, then*

$$|\{v : v \in V(G), d_G(v) = p2^k, \text{where } p \text{ is odd}\}| \text{ is even}.$$

**Proof.** Let $V(G) = \{v_1, \dots, v_n\}$ and $d_G(v_i) = q_i 2^{r_i}$, where $q_i$ is odd and $r_i \in \mathbb{N}$ $(1 \le i \le n)$. Also, let $\varphi$ be a a locally-balanced 2-partition with an open neighborhood of $G$.

Suppose, to the contrary, that $|\{v : v \in V(G), d_G(v) = p2^k, \text{where } p \text{ is odd}\}|$ is odd. Let us consider a vertex $v \in V(G)$. Since $G$ is an even graph, it is easy to see that

$$\#(v) = 0. \tag{1}$$

Let us take the sum of (1) for all vertices $v \in V(G)$. Then

$$\sum_{i=1}^{n} \#(v_i) = 0. \tag{2}$$

In this sum each vertex appears it's degree time. We can rewrite (2) as follows:

$$\sum_{i=1}^{n} d_G(v_i) \cdot \varphi^*(v_i) = 0. \tag{3}$$

Let us take out $2^k$ from the sum, we obtain

$$2^k \sum_{i=1}^{n} 2^{r_i} \cdot q_i \cdot \varphi^*(v_i) = 0. \tag{4}$$

Let us divide two sides of the equality by $2^k$. Then, we obtain

$$\sum_{i=1}^{n} 2^{r_i} \cdot q_i \cdot \varphi^*(v_i) = 0. \tag{5}$$

From (5) we obtain that the number of vertices for which $r_i = 0$ is even, which is a contradiction. □

**Corollary.** *Every* $2r$*-regular graph of odd order has no locally-balanced* 2*-partition with an open neighborhood.*

**Theorem 2.** *Let G be an odd graph and*

$$k = \min\{q : v \in V(G), d_G(v) + 1 = p2^q, \text{where } p \text{ is odd and } q \in \mathbb{N}\}.$$

*If G has a locally-balanced* 2*-partition with a closed neighborhood, then*

$$|\{v : v \in V(G), d_G(v) + 1 = p2^k, \text{where } p \text{ is odd}\}| \text{ is even}.$$

**Proof.** It can be proved using the same technique as in the proof of Theorem 1. □

Next we consider rook's graphs. For these graphs we prove the following results.

**Theorem 3.** *If* $m, n \ge 2$*, then the graph* $K_m \square K_n$ *has a locally-balanced* 2*-partition with a closed neighborhood if and only if m and n are even.*

**Proof.** Let $V(K_m \square K_n) = \{v_{ij} : 1 \le i \le m, 1 \le j \le n\}$.

First we construct a locally-balanced 2-partition with a closed neighborhood of $K_m \Box K_n$. Let us define a 2-partition $\alpha$ of $K_m \Box K_n$ as follows: for $1 \leq i \leq m$ and $1 \leq j \leq n$, let

$$\alpha(v_{ij}) = \begin{cases} \mathbf{0}, & \text{if } i+j \text{ is even,} \\ \mathbf{1}, & \text{if } i+j \text{ is odd.} \end{cases}$$

It is not difficult to see that $\alpha$ is a locally-balanced 2-partition with a closed neighbourhood of $K_m \Box K_n$.

For each 2-partition $\varphi$ of $K_m \Box K_n$, let us construct an appropriate $m \times n$ matrix $\mathbb{T} = (t_{i,j})_{m \times n}$ in the following way:

$$t_{i,j} = \varphi^*(v_{ij}).$$

Clearly, if for each $v_{ij} \in V(K_m \Box K_n)$,

$$-1 \leq \#[v_{ij}] = \sum_{k=1, k \neq j}^{n} t_{i,k} + \sum_{k=1, k \neq i}^{m} t_{k,j} + t_{i,j} \leq 1, \tag{6}$$

then $\varphi$ is a locally-balanced 2-partition with a closed neighbourhood of $K_m \Box K_n$. So, if we can construct such a matrix for which the statement (6) will be true, then we can construct an appropriate, partition which will be a locally-balanced 2-partition with a closed neighbourhood. It is easy to see, that if we have some matrix for which the statement (6) is true, then after changing some columns or rows with places the statement (6) will stay true. After this we will continue our investigation only with a matrix $\mathbb{T}$.

We now show, that if $m$ or $n$ is odd, then the graph has no locally-balanced 2-partition with a closed neighbourhood.

Suppose, to the contrary, that there exists a locally-balanced 2-partition with a closed neighbourhood $\psi$ of $K_{2m+1} \Box K_r$ $(m \geq 1, r \geq 2)$.

Let us construct a matrix $\mathbb{T} = (t_{i,j})_{(2m+1) \times r}$ with $t_{i,j} = \psi^*(v_{ij})$ and consider two cases.

**Case 1.** There is some row where all elements have the same sign.

Without loss of generality we may assume that this is the first row and the value of all elements is 1. Let us consider the vertex $v_{1j}$. Clearly,

$$-1 \leq \#[v_{1j}] = 2m + 1 + \sum_{i=2}^{r} t_{i,j} \leq 1.$$

From this and taking into account that $2m+1 \geq 3$, we obtain

$$\sum_{i=2}^{r} t_{i,j} < -1 \quad \forall j = \overline{1, 2m+1}. \tag{7}$$

Let us consider $(2m+1)$-th column, from (7) we obtain

$$\sum_{i=1}^{r} t_{i,2m+1} < 0. \tag{8}$$

From (8) and taking $j = 2m+1$ in (6), we have

$$\sum_{j=1}^{2m} t_{i,j} \geq 0 \quad \forall i = \overline{1, r}. \tag{9}$$

Let us sum (7) with all $j = \overline{1, 2m}$. We obtain

$$\sum_{j=1}^{2m}\sum_{i=2}^{r} t_{i,j} < 0. \tag{10}$$

Let us sum (9) with all $i = \overline{2,r}$. We obtain

$$\sum_{i=2}^{r}\sum_{j=1}^{2m} t_{i,j} \geq 0, \tag{11}$$

which is a contradiction.

**Case 2.** Does not exist a row, where all elements have the same sign.

Without loss of generality we may assume that in the first row the number of 1's is greater than the number of $-1$'s. We have

$$\sum_{j=1}^{2m+1} t_{1,j} > 0. \tag{12}$$

From (6) and (12) we obtain

$$\sum_{i=2}^{r} t_{i,j} \leq 0 \quad \forall j = \overline{1,2m+1}. \tag{13}$$

There is an element with a value $-1$ in the first row. We can move that column to the end. Taking into account that $t_{1,2m+1} = -1$ and from (13), we obtain

$$\sum_{i=1}^{r} t_{i,2m+1} < 0. \tag{14}$$

From (6) and (14 we have

$$\sum_{j=1}^{2m} t_{i,j} \geq 0 \quad \forall i = \overline{1,r}. \tag{15}$$

Let us sum (13) over all $j = \overline{1,2m}$. We obtain

$$\sum_{j=1}^{2m}\sum_{i=2}^{r} t_{i,j} \leq 0. \tag{16}$$

Let us sum (15) over all $i = \overline{2,r}$. We obtain

$$\sum_{i=2}^{r}\sum_{j=1}^{2m} t_{i,j} \geq 0. \tag{17}$$

If $r$ is even, then we have the strict inequalities in (13) and (16), which is a contradiction. It means that $r$ is odd and $r = 2l+1$ for some $l \in \mathbb{N}$.

It is easy to see that (16) and (17) will be true if and only if there are equalities in both statements and that will be if and only if there are equalities in (13) and (15). It means that number of 1's and $-1$'s are equal in all rows or columns not taking into account the first row and the last column. Hence,

$$\sum_{i=2}^{r} t_{i,j} = 0 \quad \forall j = \overline{1,2m}, \tag{18}$$

$$\sum_{j=1}^{2m} t_{i,j} = 0 \quad \forall i = \overline{2,r}. \tag{19}$$

Let us note that each element of the last column cannot be $-1$; otherwise, by transposing the matrix $\mathbb{T}$, we obtain a new matrix $\mathbb{T}'$ with an odd number of rows, where all elements of the last row have the same sign, which contradicts Case 1. So, we may assume that there is an element with value 1 in the last column. Let $t_{i_0,2m+1}$ be this element. Clearly $i_0 > 1$. Let us rearrange the columns of the matrix $\mathbb{T}$ to have 1-valued entries of the first row at the beginning of row. We will not change the place of the last column. Then, we obtain the following matrix:

$$\begin{pmatrix} 1 & 1 & \dots & 1 & -1 & \dots & -1 & -1 \\ t_{2,1} & t_{2,2} & \cdots & t_{2,j} & t_{2,j+1} & \cdots & t_{2,2m} & t_{2,2m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{i_0-1,1} & t_{i_0-1,2} & \cdots & t_{i_0-1,j} & t_{i_0-1,j+1} & \cdots & t_{i_0-1,2m} & t_{i_0-1,2m+1} \\ t_{i_0,1} & t_{i_0,2} & \cdots & t_{i_0,j} & t_{i_0,j+1} & \cdots & t_{i_0,2m} & 1 \\ t_{i_0+1,1} & t_{i_0+1,2} & \cdots & t_{i_0+1,j} & t_{i_0+1,j+1} & \cdots & t_{i_0+1,2m} & t_{i_0+1,2m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{2l+1,1} & t_{2l+1,2} & \cdots & t_{2l+1,j} & t_{2l+1,j+1} & \cdots & t_{2l+1,2m} & t_{2l+1,2m+1} \end{pmatrix}.$$

Let us consider two cases.

**Subcase 2A:** $t_{i_0,1} = -1$.

We calculate $\#[v_{i_0 1}]$ by taking into account (18) and (19), we obtain

$$\#[v_{i_0 1}] = t_{1,1} + t_{i_0,2m+1} + \sum_{j=1}^{2m} t_{i_0,j} + \sum_{k=2}^{2l+1} t_{k,1} - t_{i_0,1} = 1 + 1 + m - m + l - l + 1 = 3,$$

which is a contradiction.

**Sabcase 2B:** $t_{i_0,1} = 1$.

Using the same technique as in Subcase 2A, we obtain that $t_{i_0,2} = t_{i_0,3} = \dots = t_{i_0,j} = 1$, where $j$ is the last column, which $t_{1,j} = 1$. Now our matrix looks like:

$$\begin{matrix} t_{i_0,2} = t_{1,2} & t_{i_0,3} = t_{1,3} & \dots & t_{i_0,j} = t_{1,j} \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & -1 & \dots & -1 & -1 \\ t_{2,1} & t_{2,2} & \cdots & t_{2,j} & t_{2,j+1} & \cdots & t_{2,2m} & t_{2,2m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{i_0-1,1} & t_{i_0-1,2} & \cdots & t_{i_0-1,j} & t_{i_0-1,j+1} & \cdots & t_{i_0-1,2m} & t_{i_0-1,2m+1} \\ 1 & 1 & \cdots & 1 & t_{i_0,j+1} & \cdots & t_{i_0,2m} & 1 \\ t_{i_0+1,1} & t_{i_0+1,2} & \cdots & t_{i_0+1,j} & t_{i_0+1,j+1} & \cdots & t_{i_0+1,2m} & t_{i_0+1,2m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{2l+1,1} & t_{2l+1,2} & \cdots & t_{2l+1,j} & t_{2l+1,j+1} & \cdots & t_{2l+1,2m} & t_{2l+1,2m+1} \end{pmatrix}.$$

By (12) and taking into account that $t_{1,2m+1} = -1$, we have

$$\sum_{j=1}^{2m} t_{1,j} > 0.$$

Clearly,

$$|\{\ j\ :\ t_{1,j} = 1,\ j = \overline{1,2m}\}| > m.$$

From this and taking into account that $t_{i_0,2} = t_{1,2}, t_{i_0,3} = t_{1,3}, \dots, t_{i_0,j} = t_{1,j}$, we obtain

$$|\{\ j\ :\ t_{i_0,j} = 1,\ j = \overline{1,2m}\}| > m.$$

which contradicts (19). □

***Theorem 4.*** *If $m,n > 2$ and either $m$ and $n$ are odd or $m$ and $n$ are even, then the graph $K_m \Box K_n$ has no locally-balanced 2-partition with an open neighborhood.*

***Proof.*** If $m$ and $n$ are odd, then, by Corollary , $K_m \Box K_n$ has no locally-balanced 2-partition with an open neighborhood.

Let us consider the case when $m$ and $n$ are even and $m,n > 2$. Suppose, to the contrary, that there exists a locally-balanced 2-partition with an open neighborhood $\varphi$ of $K_m \Box K_n$. Since $K_m \Box K_n$ is Eulerian, we have

$$\#(v_{ij}) = 0 \qquad \forall i = \overline{1,m}, \quad \forall j = \overline{1,n}. \tag{20}$$

Let us sum (20) over all values of $i$ and $j$ we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \#(v_{ij}) = 0. \tag{21}$$

In this sum each vertex appears it's degree time. We can rewrite (21) as follows:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \varphi^*(v_{ij}) \cdot (m+n-2) = 0.$$

Since $m+n-2 > 0$, we obtain

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \varphi^*(v_{ij}) = 0. \tag{22}$$

Let us now consider $\#(v_{ij})$:

$$\#(v_{ij}) = \sum_{k=1, k\neq j}^{n} \varphi^*(v_{ik}) + \sum_{k=1, k\neq i}^{m} \varphi^*(v_{kj}) = 0 \qquad \forall i = \overline{1,m}, \quad \forall j = \overline{1,n}. \tag{23}$$

Let us sum (23) over all $j = \overline{1,n}$, we have

$$\begin{gathered} \sum_{k=1}^{n} \varphi^*(v_{ik}) \cdot (n-1) + \sum_{j=1}^{n} \sum_{k=1, k\neq i}^{m} \varphi^*(v_{kj}) = 0 \qquad \forall i = \overline{1,m}. \\ \sum_{k=1}^{n} \varphi^*(v_{ik}) \cdot (n-2) + \sum_{j=1}^{n} \sum_{k=1}^{m} \varphi^*(v_{kj}) = 0 \qquad \forall i = \overline{1,m}. \end{gathered} \tag{24}$$

By (24) and taking into account (22), we obtain

$$\sum_{k=1}^{n} \varphi^*(v_{ik}) \cdot (n-2) = 0 \qquad \forall i = \overline{1,m}.$$

Since $n-2 > 0$, we have

$$\sum_{k=1}^{n} \varphi^*(v_{ik}) = 0 \qquad \forall i = \overline{1,m}. \tag{25}$$

Using the same technique and taking sum of (23) over all $i = \overline{1,m}$, we obtain

$$\sum_{k=1}^{m} \varphi^*(v_{kj}) = 0 \qquad \forall j = \overline{1,n}. \tag{26}$$

Without loss of generality we may assume that $\varphi^*(v_{11}) = 1$. By (25), we have

$$\sum_{i=2}^{m} \varphi^*(v_{i1}) + \varphi^*(v_{11}) = 0, \qquad \sum_{i=2}^{m} \varphi^*(v_{i1}) = -1. \tag{27}$$

By (26), we have

$$\sum_{j=2}^{n} \varphi^*(v_{1j}) + \varphi^*(v_{11}) = 0, \qquad \sum_{j=2}^{n} \varphi^*(v_{1j}) = -1. \tag{28}$$

Let us calculate $\#(v_{11})$, taking into account (27) and (28),

$$\#(v_{11}) = \sum_{i=2}^{m} \varphi^*(v_{i1}) + \sum_{j=2}^{n} \varphi^*(v_{1j}) = -2.$$

The latter contradicts to (20). □

Finally, we consider locally-balanced 2-partitions of cycles of powers. First of all, let us note that if $n$ is odd, then $C_n^k$ is a $2k$-regular graph of odd order $n$, hence, by Corollary, $C_n^k$ has no locally-balanced 2-partition with an open neighborhood. On the other hand, the following results hold.

***Proposition*** . *If $n$ and ($k$ or $\frac{n}{k+1}$) are even ($n,k \in \mathbb{N}$), then $C_n^k$ has a locally-balanced 2-partition with an open neighborhood.*

***Proof.*** Let us consider two cases.

**Case 1:** $n$ and $k$ are even.

For the proof of this case, we define a 2-partition $\lambda$ of $C_n^k$ as follows: for $0 \le i \le n-1$, let

$$\lambda(v_i) = \begin{cases} \mathbf{0}, & \text{if } i \text{ is even,} \\ \mathbf{1}, & \text{if } i \text{ is odd.} \end{cases}$$

It is not difficult to see that $\lambda$ is a locally-balanced 2-partition of $C_n^k$ with an open neighborhood.

**Case 2:** $n$ and $\dfrac{n}{k+1}$ are even.

In this case we define a 2-partition $\psi$ of $C_n^k$ as follows: we color $v_0, v_1, \dots, v_{n-1}$ vertices sequentially, coloring the first $k+1$ vertices by **0**, then the next $k+1$ vertices by **1** and so on. It is easy to verify that $\psi$ is a locally-balanced 2-partition of $C_n^k$ with an open neighborhood. □

It is easy to see that the 2-partition $\lambda$ constructed in the proof of Proposition is also a locally-balanced 2-partition with a closed neighborhood of $C_n^k$.

***Theorem 5.*** *If $n$ is even, $k$ is odd and $\dfrac{lcm(n,k+1)}{k+1}$ is odd ($n,k \in \mathbb{N}$), then $C_n^k$ has no locally-balanced 2-partition with an open neighborhood.*

***Proof.*** Suppose, to the contrary, that there exists a locally-balanced 2-partition with an open neighborhood $\varphi$ of the graph $C_n^k$. Let as consider the following sum

$$\sum_{i=0}^{k-1} \varphi^*(v_i) = t. \tag{29}$$

Clearly, $t \neq 0$ ($k$ is odd). Since $\varphi$ is a locally-balanced 2-partition with an open neighborhood of $C_n^k$, we have

$$\#(v_k) = \sum_{i=0}^{k-1} \varphi^*(v_i) + \sum_{i=k+1}^{2k} \varphi^*(v_i) = 0. \tag{30}$$

For $i \in \mathbb{Z}_{\geq 0}$, we define an auxiliary function $f(i)$ as follows:

$$f(i) = \sum_{j=0}^{k-1} \varphi^*(v_{((i+j) \bmod n)}).$$

By (30) we have

$$\#(v_k) = f(0) + f(k+1) = 0,$$
$$\#(v_{(2k+1) \bmod n}) = f(k+1) + f(2 \cdot (k+1)) = 0,$$
$$\vdots$$
$$\#(v_{(r(k+1)-1) \bmod n}) = f((r-1) \cdot (k+1)) + f(r \cdot (k+1)) = 0.$$

This implies that

$$f(0) = -f(k+1),$$
$$f(k+1) = -f(2 \cdot (k+1)),$$
$$\vdots$$
$$f((r-1) \cdot (k+1)) = -f(r \cdot (k+1)).$$

Using this, we can write the following statement:

$$f(0) = l \cdot f(a \cdot (k+1)),$$
$$\text{where} \quad l = \begin{cases} 1, & \text{if } a \text{ is even} \\ -1, & \text{if } a \text{ is odd.} \end{cases} \tag{31}$$

By (31) and taking into account that $\dfrac{lcm(n,k+1)}{k+1}$ is odd, we have

$$f(0) = -f\left(\left(\frac{lcm(n,k+1)}{k+1}\right) \cdot (k+1)\right) = -f(lcm(n,k+1)).$$

From this and taking into account that $lcm(n,k+1) \bmod n = 0$, we have

$$\sum_{j=0}^{k-1} \varphi^*(v_{(j \bmod n)}) = f(0) = -f(lcm(n,k+1)) =$$
$$= -\sum_{j=0}^{k-1} \varphi^*(v_{((j+lcm(n,k+1)) \bmod n)}) = -\sum_{j=0}^{k-1} \varphi^*(v_{(j \bmod n)}).$$

This implies that $\sum_{j=0}^{k-1} \varphi^*(v_{(j \bmod n)}) = 0$, which contradicts (29). □

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## ԳՐԱՖՆԵՐԻ ՈՐՈՇ ԴԱՍԵՐԻ ԼՈԿԱԼ-ՀԱՎԱՍԱՐԱԿՇՌՎԱԾ 2-ՏՐՈՀՈՒՄՆԵՐԻ ՄԱՍԻՆ

Այս աշխատանքում ստացվել են անհրաժեշտ, բավարար պայմաններ գրաֆների որոշ դասերի լոկալ-հավասարակշռված 2-տրոհումների գոյության համար բաց (փակ) շրջակայքով: Տրվում են անհրաժեշտ պայմաններ կենտ և զույգ գրաֆների լոկալ-հավասարակշռված 2-տրոհումների գոյության համար: Ստացվել են նաև որոշ արդյունքներ նավակների գրաֆների և ցիկլերի աստիճանների լոկալ-հավասարակշռված 2-տրոհումների գոյության համար: Ապացուցվել է, որ եթե $m,n \geq 2$, ապա $K_m \Box K_n$ նավակների գրաֆն ունի լոկալ-հավասարակշռված 2-տրոհում փակ շրջակայքով այն և միայն այն դեպքում, եթե $m$-ը և $n$-ը զույգ են և կառուցվում են պահանջվող 2-տրոհումները բազմանդամային բարդություն ունեցող ալգորիթմների միջոցով:

А. Г. ГАРИБЯН

## О ЛОКАЛЬНО-СБАЛАНСИРОВАННЫХ 2-РАЗБИЕНИЯХ НЕКОТОРЫХ КЛАССОВ ГРАФОВ

В настоящей работе даются необходимые и достаточные условия существования локально-сбалансированных 2-разбиений с открытой (закрытой) окрестностью для некоторых классов графов. В частности, в работе даны необходимые условия существования локально-сбалансированных 2-разбиений четных и нечетных графов. В работе также получены некоторые результаты существования локально-сбалансированных 2-разбиений ладейных графов и различных степеней цикла. В частности, в работе доказано, что если $m,n \geq 2$, то ладейный граф $K_m \square K_n$ имеет локально-сбалансированное 2-разбиение с закрытой окрестностью тогда и только тогда, когда $m$ и $n$ – четные числа. Кроме того, все предложенные доказательства являются конструктивными и строят требуемые 2-разбиения с помощью полиномиальных алгоритмов.