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Mathematics

ON CONSTANT COEFFICIENT PDE SYSTEMS AND INTERSECTION MULTIPLICITIES

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In this paper we consider the concept of the multiplicity of intersection points of plane algebraic curves $p,q=0$, based on partial differential operators. We evaluate the exact number of maximal linearly independent differential conditions of degree k for all $k \geq 0$. On the other hand, this gives the exact number of maximal linearly independent polynomial and polynomial-exponential solutions, of a given degree k, for homogeneous PDE system $p(D)f=0$, $q(D)f=0$.

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Introduction. The space of all polynomials in two variables is denoted by Π. The subspace of polynomials of total degree at most m is denoted by Π_m. The two variables are denoted by $\mathbf{x}=(x_1,x_2)$ or sometimes by (x,y). For $\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2$ set $\alpha!=\alpha_1!\,\alpha_2!,\ \ |\alpha|=\alpha_1+\alpha_2$.

Then, for $\mathbf{x}=(x_1,x_2)$ and $\mathbf{y}=(y_1,y_2)$ denote

$$\mathbf{xy}=x_1y_1+x_2y_2,\ \ \mathbf{x}^\alpha:={x_1}^{\alpha_1}{x_2}^{\alpha_2}.$$

The differential operator given by the polynomial $r\in\Pi$ is denoted by

$$r(D):=r(D_1,D_2),\ r^{(\alpha)}:=D^\alpha r:=(D_1)^{\alpha_1}(D_2)^{\alpha_2}r,\ \text{where } D_i:=D_{x_i}.$$

To simplify the notation, we shall use the same letter p, say, to denote the polynomial p and the curve given by the equation $p(x,y)=0$. Thus the notation $\lambda\in p$ means that the point λ belongs to the curve $p(x,y)=0$. Similarly $p\cap q$ for polynomials p and q stands for the set of intersection points of the curves $p(x,y)=0$ and $q(x,y)=0$.

Next we bring the *PD multiplicity space* (see [1–3]) for $\lambda\in r\in\Pi$:

$$\mathcal{M}_\lambda(r)=\left\{h\in\Pi : D^\alpha h(D)r(\lambda)=0\ \forall\alpha\in\mathbb{Z}_+^2\right\}.$$

We have that (see [2]) the space $\mathcal{M}_\lambda(r)$ is D-invariant, meaning that

$$f\in\mathcal{M}_\lambda(r)\Rightarrow\frac{\partial f}{\partial x}\text{ and }\frac{\partial f}{\partial y}\in\mathcal{M}_\lambda(r). \tag{1}$$

Denote by $\mathcal{Z}_0=p\cap q$ the set of intersection points of curves $p,q\in\Pi$.

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Definition. Suppose that $p,q \in \Pi$ and $\lambda \in \mathcal{Z}_0$. Then the following space is called the multiplicity space of the intersection point λ :

$$\mathcal{M}_\lambda(p,q) = \mathcal{M}_\lambda(p) \cap \mathcal{M}_\lambda(q).$$

The number $\mu_\lambda(p,q) := \dim \mathcal{M}_\lambda(p,q)$ is called the arithmetical multiplicity of the point λ.

Let

$$p = \sum_{i+j=m} a_{ij} x^i y^j, \quad q = \sum_{i+j=n} b_{ij} x^i y^j. \tag{2}$$

In the sequel we will use Resultant of p and q (see [4], section 10):

$$R(p,q) = \begin{vmatrix} a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_m & & & \\ & a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_m & & \\ & & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \ddots & \\ & & & a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_m \\ b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & b_n & & & \\ & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & b_n & & \\ & & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \ddots & \\ & & & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & b_n \end{vmatrix}.$$

Here we have n rows of a's and m rows of b's. All other entries equal 0.

Theorem 1. (see [4], Theorem 10.7). *The homogeneous polynomials p and q given in (2) have no common factor if and only if $R(p,q) \neq 0$.*

Intersection Multiplicity as PDE System Solution. Let us start with the following result (see also Theorem 6 in [5]):

Theorem 2. (see [2], Theorem 5). *Suppose that λ is a solution of an algebraic equation $r(\mathbf{x}) = 0,\ r \in \Pi$. Then a polynomial $h \in \Pi$ belongs to the multiplicity space $\mathcal{M}_\lambda(r)$ if and only if the function*

$$y = h(\mathbf{x}) exp(\lambda \mathbf{x})$$

is a solution of the PDE

$$r(D)y = 0. \tag{3}$$

In particular, for $\lambda = \theta := (0,0)$ the following relation holds

$$h \in \mathcal{M}_\theta(r) \iff r(D)h = 0, \text{where } r,h \in \Pi.$$

Denote the space of polynomial-exponential solutions of PDE (3) by

$$\mathcal{S}_\lambda(r) := \{y = h(\mathbf{x}) exp(\lambda \mathbf{x}) : r(D)y = 0,\ h \in \Pi\}.$$

For $p,q \in \Pi$, consider the following PDE system:

$$\begin{cases} p(D)f = 0, \\ q(D)f = 0. \end{cases} \tag{4}$$

The corresponding space of solutions for PDE system denote by

$$\mathcal{S}_\lambda(p,q) := \mathcal{S}_\lambda(p) \cap \mathcal{S}_\lambda(q).$$

Main Result. Denote for $\lambda \in r \in \Pi$:

$$\mathcal{M}_{k,\lambda}(r) := \mathcal{M}_{\lambda}(r) \cap \Pi_k, \quad \mathcal{S}_{k,\lambda}(r) := \mathcal{S}_{\lambda}(r) \cap \Pi_k.$$

First, we are going to find the dimensions of these spaces for any k.

Of course, in view of Theorem 2 the following equality holds

$$\dim \mathcal{M}_{k,\lambda}(r) = \dim \mathcal{S}_{k,\lambda}(r), \quad r \in \Pi. \tag{5}$$

We say that λ is an m_0-fold zero for p, if the least nonzero homogenous part of $p(\mathbf{x}+\lambda)$ is the m_0-homogeneous part.

Theorem 3. *Suppose that $p \in \Pi$ is a polynomial, for which λ is an m_0-fold zero. Then the PD equation $p(D)f = 0$ has exactly D_k linearly independent solutions of the form*

$$h(\mathbf{x})exp(\lambda \mathbf{x}), \quad h \in \Pi_k,$$

where D_k is the k^{th} partial sum of the following series:

$$\sum_{i=0}^{\infty} d_i := 1 + 2 + \cdots + m_0 + m_0 + \cdots + m_0 + \cdots \tag{6}$$

In view of (5), we obtain

Corollary 1. *Suppose that $p \in \Pi$ is a polynomial, for which λ is an m_0-fold zero. Then there are exactly D_k linearly independent polynomials in the space $\mathcal{M}_{k,\lambda}(p)$, where D_k is the k^{th} partial sum of the series (6).*

Proof of Theorem 3. Without loss of generality assume that

$$\lambda = \theta := (0,0) \in p.$$

Suppose $f \in \Pi_m$,

$$f(x,y) = \sum_{i+j \le m} \gamma_{ij} x^i y^j.$$

Denote the k^{th} homogeneous part of f by f_k, i.e.

$$f_k(x,y) = \sum_{i+j=k} \gamma_{ij} x^i y^j.$$

Suppose that p is a bivariate polynomial of degree m_1 having m_0 fold zero at the origin:

$$p(x,y) = \sum_{m_0 \le i+j \le m_1} a_{ij} x^i y^j.$$

For the brevity set $m = m_0$. Evidently we have that $\mathcal{S}_{k,\theta}(p) = \Pi_k$ if $k \le m-1$.

Now consider the space $\mathcal{S}_{k,\theta}(p)$, where $k = m+s,\ s \ge 0$. We have $f \in \mathcal{S}_{k,\theta}(p)$ if and only if

$$p(D)f = (p_m + p_{m+1} + \cdots + p_{m+s})(D)(f_m + \cdots + f_{m+s}) = 0,$$

i.e.

$$\begin{aligned}
p_m(D)f_m + p_{m+1}(D)f_{m+1} + p_{m+2}(D)f_{m+2} + \cdots &+ p_{m+s}(D)f_{m+s} \\
+ \quad p_m(D)f_{m+1} + p_{m+1}(D)f_{m+2} + \cdots &+ p_{m+s-1}(D)f_{m+s} \\
+ \quad \cdots &+ \\
+ p_m(D)f_{m+s-1} &+ p_{m+1}(D)f_{m+s} \\
&+ p_m(D)f_{m+s} = 0.
\end{aligned}$$

The coefficient of $x^\alpha y^\beta$, where $\alpha+\beta=r,\ r=0,\dots s$, is obtained from the r^{th} line in above and equals to zero:

$$\sum_{i+j=m} a_{ij}\frac{(i+\alpha)!}{\alpha!}\frac{(j+\beta)!}{\beta!}\gamma_{i+\alpha j+\beta}+\sum_{k=1}^{s-r}\sum_{i+j=m+k} a_{ij}\frac{(i+\alpha)!}{\alpha!}\frac{(j+\beta)!}{\beta!}\gamma_{i+\alpha j+\beta}=0,$$

for $\forall\alpha,\beta$ with $\alpha+\beta=r,\ r=0,\dots,k-m$. Here we separated the first sum for the convenience.

By multiplying above equality by $\alpha!\beta!$, we get

$$\sum_{i+j=m} a_{ij}(i+\alpha)!(j+\beta)!\gamma_{i+\alpha j+\beta}+\sum_{s=1}^{k-m-r}\sum_{i+j=m+s} a_{ij}(i+\alpha)!(j+\beta)!\gamma_{i+\alpha j+\beta}=0,$$

for $\forall\alpha,\beta$ with $\alpha+\beta=r,\ r=0,\dots,k-m$. Variables present in the first sum are $\gamma_{0m+r},\gamma_{1m+r-1},\dots,\gamma_{mr},\gamma_{m+1r-1},\dots,\gamma_{m+r0}$. The corresponding main matrix is

$$\begin{pmatrix}
a_0c_0 & a_1c_1 & \cdots & \cdots & \cdots & \cdots & a_mc_m & 0 & \cdots & \cdots & 0\\
0 & a_0c_1 & a_1c_2 & \cdots & \cdots & \cdots & \cdots & a_mc_{m+1} & 0 & \cdots & 0\\
0 & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & 0\\
0 & \cdots & 0 & a_0c_{k-m-1} & a_1c_{k-m} & \cdots & \cdots & \cdots & \cdots & a_mc_{k-1} & 0\\
0 & \cdots & \cdots & 0 & a_0c_{k-m} & a_1c_{k-m+1} & \cdots & \cdots & \cdots & \cdots & a_mc_k
\end{pmatrix},$$

where we set for the brevity $a_i=a_{im-i}$ and $c_i=i!(m-i+r)!$.

By dividing the i^{th} column by c_i, we get the following $(k-m+1)\times k$ matrix

$$\begin{pmatrix}
a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_m & 0 & \cdots & \cdots & 0\\
0 & a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_m & 0 & \cdots & 0\\
0 & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & 0\\
0 & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_m & 0\\
0 & \cdots & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_m
\end{pmatrix}.$$

Since $p_m\neq 0$, in view of Theorem 1, we get that the above matrix is full (row) rank. Thus we get that all $1+2+\cdots+(s+1)$ conditions are independent and

$$\begin{aligned}
\dim\mathcal{S}_{k,\theta}(p)&=\dim\Pi_k-[1+\cdots+(s+1)]=[1+\cdots+(k+1)]-[1+\cdots+(s+1)]\\
&=[1+\cdots+m]+[(m+1)+\cdots+(m+s+1)]-[1+2+\cdots+(s+1)]\\
&=\dim\Pi_{m-1}+m(s+1).
\end{aligned}$$

Thus we obtain that $\dim\mathcal{S}_{k,\theta}(p)=1+\cdots+(m-1)+m(k-m+2)$, where $k\geq m-1$. □

For the next result we accept a very common restriction from the theory of intersection. Namely, we assume that the two polynomials p and q have no common tangent line at an intersection point $\lambda\in\mathcal{Z}_0$. This means that the lowest homogeneous parts of the polynomials $p(\mathbf{x}+\lambda)$ and $q(\mathbf{x}+\lambda)$ have no common factor.

Theorem 4. *Suppose that polynomials $p,q\in\Pi$ have no common tangent line at an intersection point $\lambda\in\mathcal{Z}_0$. Suppose also that for p and q the point λ is an m_0 and n_0-fold zero, respectively, $m_0\le n_0$. Then the PDE system has exactly D_k linearly independent solutions of the form*

$$h(\mathbf{x})exp(\lambda\mathbf{x}),\ \ h\in\Pi_k,$$

where D_k is the k^{th} partial sum of the following series:

$$\sum_{i=0}^{\infty}d_i:=1+2+\cdots+(m_0-1)+\underbrace{m_0+\cdots+m_0}_{n_0-m_0+1}+(m_0-1)+\cdots+1+0+\cdots+0+\cdots. \tag{7}$$

Corollary 2. *Suppose that polynomials $p,q\in\Pi$ have no common tangent line at an intersection point $\lambda\in\mathcal{Z}_0$. Suppose also that for p and q the point λ is an m_0 and n_0-fold zero, respectively, $m_0\le n_0$. Then there are exactly D_k linearly independent polynomials in the space $\mathcal{M}_{k,\lambda}(p,q)$, where D_k is the k^{th} partial sum of the series* (7).

Proof of Theorem 4. Without loss of generality assume that

$$\lambda=\theta:=(0,0)\in\mathcal{Z}_0.$$

Suppose that p and q are bivariate polynomials of degree m_1 and n_1 having m_0 and n_0-fold zero at the origin, respectively:

$$p(x,y)=\sum_{m_0\le i+j\le m_1}a_{ij}x^iy^j,\quad q(x,y)=\sum_{n_0\le i+j\le n_1}b_{ij}x^iy^j.$$

For brevity set $m=m_0$ and $n=n_0$. Suppose that $m\le n$. Let $f\in\Pi:$

$$f(x,y)=\sum_{i+j}\gamma_{ij}x^iy^j.$$

Note that

$$p_l(D)f_{l+s}=\sum_{\alpha+\beta=s}\sum_{i+j=l}a_{ij}\frac{(i+\alpha)!}{\alpha!}\frac{(j+\beta)!}{\beta!}\gamma_{i+\alpha j+\beta}x^{s-\alpha}y^{s-\beta}.$$

Let

$$\mathcal{S}_k(p,q):=\mathcal{S}_\theta(p,q)\cap\Pi_k.$$

Evidently we have that $\mathcal{S}_{k,\theta}(p,q)=\mathcal{S}_{k,\theta}(p)$, if $k\le n-1$.

Consider the case $k=n+s,\ s\ge 0$. We have that $f\in\mathcal{S}_{n+s}(p,q)$ if and only if

$$p(D)f=[p_m+p_{m+1}+\cdots+p_{n+s}](D)(f_m+f_{m+1}+\cdots+f_{n+s})=0$$

and

$$q(D)f=[q_n+q_{n+1}+\cdots+q_{n+s}](D)(f_n+f_{n+1}+\cdots+f_{n+s})=0.$$

Thus, to the obtained conditions for p, we add the following $1+\cdots+(s+1)$ for $q:$

$$\begin{array}{lllll}
q_n(D)f_n+q_{n+1}(D)f_{n+1}+q_{n+2}(D)f_{n+2}+ & \cdots & + & q_{n+s}(D)f_{n+s} \\
\quad +\ q_n(D)f_{n+1}+q_{n+1}(D)f_{n+2}+ & \cdots & + & q_{n+s-1}(D)f_{n+s} \\
\qquad\qquad + & \cdots & + & \\
\qquad\qquad +q_n(D)f_{n+s-1}+ & & & q_{n+1}(D)f_{n+s} \\
 & & + & q_n(D)f_{n+s}=0\,.
\end{array}$$

$$\sum_{i+j=n} b_{ij} \frac{(i+\alpha)!}{\alpha!} \frac{(j+\beta)!}{\beta!} \gamma_{i+\alpha j+\beta} + \sum_{k=1}^{s-r} \sum_{i+j=n+k} b_{ij} \frac{(i+\alpha)!}{\alpha!} \frac{(j+\beta)!}{\beta!} \gamma_{i+\alpha j+\beta} = 0$$

for $\forall \alpha, \beta$, with $\alpha+\beta = r$, $r = 0, \ldots s$. In the same way as in the proof of the previous theorem, by using Theorem 1, we get that all these conditions are independent together with the described conditions for p. Thus we get

$$\begin{aligned} \dim \mathcal{S}_{n+s}(p,q) &= \dim \mathcal{S}_{n+s}(p) - (1+2+\cdots+s+(s+1)) \\ &= (1+2+\cdots+(m-1)) + m(n+s+2-m) - (1+2+\cdots+s+(s+1)) \\ &= (1+2+\cdots+(m-1)) + m(n-m+1) + (m-1) + (m-2) + \cdots + (m-s-1). \end{aligned}$$

In particular, for $s = m-2$ we get that

$$\dim \mathcal{S}_{m+n-2}(p,q) = m(m-1) + m(n-m+1) = nm.$$

In the case of $s = m-1$ we get that

$$\dim \mathcal{S}_{m+n-1}(p,q) = m(m-1) + m(n-m+1) + 0 = nm.$$

This means that there is no polynomial of degree $m+n-1$ in $\mathcal{S}_{m+n-1}(p,q)$. Hence in view of D-invariance we conclude readily that there is no polynomial of degree $\geq m+n-1$ in $\mathcal{S}_{\theta}(p,q)$. □

In particular, we obtained a result of Avagyan (Theorem 3 in [6]) stating that $\dim \mathcal{S}_{\lambda}(p,q) = m_0 n_0$, if the conditions of Theorem 4 hold.

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Ն. Կ. ՎԱՐԴԱՆՅԱՆ

ՀԱՍՏԱՏՈՒՆ ԳՈՐԾԱԿԻՑՆԵՐՈՎ ՄԱՍՆԱԿԱՆ ԱԾԱՆՑՅԱԼՆԵՐՈՎ ՀԱՄԱԿԱՐԳԵՐԻ ԵՎ ԿՈՐԵՐԻ ՀԱՏՈՒՄՆԵՐԻ ՊԱՏԻԿՈՒԹՅԱՆ ՎԵՐԱԲԵՐՅԱԼ

Այս հոդվածում քննարկվում է հարթ հանրահաշվական $p,q=0$, կորերի հատման կետերի պատիկության հասկացությունը՝ հիմնված մասնակի ածանցյալներով տրվող օպերատորների վրա: Մենք որոշում ենք k աստիճանի առավելագույն գծորեն անկախ դիֆերենցիալ պայմանների ճշգրիտ քանակը բոլոր ոչ բացասական k-երի համար: Մյուս կողմից սա տալիս է $p(D)f=0$, $q(D)f=0$ մասնական ածանցյալներով համասեռ հավասարումների համակարգի k աստիճանի մաքսիմալ գծորեն անկախ բազմանդամային և էքսպոնենցիալ-բազմանդամային լուծումների ճշգրիտ քանակը:

Н. К. ВАРДАНЯН

О СИСТЕМАХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ С ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ И КРАТНОСТЯХ ПЕРЕСЕЧЕНИЙ КРИВЫХ

В этой статье рассматривается понятие кратности точек пересечения плоских алгебраических кривых $p,q=0$ на основе операторов в частных производных. Мы определяем точное число максимальных линейно независимых дифференциальных условий степени k для всех неотрицательных k. С другой стороны, это дает точное число максимальных линейно независимых полиномиальных и полиномиально-экспоненциальных решений заданной степени k для однородной системы уравнений в частных производных $p(D)f=0$, $q(D)f=0$.