

*Mechanics*

ON THE OPTIMAL STABILIZATION OF A DOUBLE MATHEMATICAL  
PENDULUM HAVING A MOVABLE SUSPENSION CENTER

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The problem of optimal stabilization of a double pendulum, when its suspension center moved in the horizontal direction according to the given law, has been treated. The problem was reduced to the case of a linear nonuniform system that was solved in the event when the first and the second pendulums had equal masses and lengths. An optimal Lyapunov function and an optimal control action have been constructed.

**Keywords:** driven double pendulum, optimal stabilization, equations of motion, Lagrange's equations, Lyapunov function, Lyapunov–Bellman method.

**Introduction.** The problems of stabilization and control of a mathematical pendulum having a movable suspension center were the subjects of study of many researchers at different times. As early in 19<sup>th</sup> century, English researcher J. Reyleigh [1] has shown, that the lower stable equilibrium position of a pendulum might become unstable, when its suspension center oscillated at a specific frequency. An English mathematician In Stephenson proved [2], that it was possible to keep the pendulum at the upper equilibrium position, if its suspension center is oscillated in the vertical direction. Then, a comprehensive study of this problem was conducted by P.L. Kapitsa [3]. The investigation of driven mathematical pendulum is of importance also nowadays, as was dealt with in [4–6]. In [4] the problem of double pendulum stability has been studied for four possible equilibrium positions, the suspension point of the first pendulum being oscillated at high frequency in the vertical direction. A comprehensive analysis of studies concerning the problems of stability, control and stabilization of driven pendulum is given in [6]. The stabilization problem of a driven single pendulum, when the suspension center moved in the horizontal direction, was studied in [7].

In the present paper the optimal stabilization problem of a driven double pendulum is considered, when the upper suspension center is moved in the horizontal direction according to a given law.

**Problem Statement and Equations of Motion.** Suppose there is a double pendulum composed of  $l_1$  and  $l_2$  long lightweight rods holding material particles  $M_1$  and  $M_2$  of  $m_1$  and  $m_2$  masses respectively. Now assume, that upper pendulum may swing about the horizontal axis  $Oz$  (that is perpendicular to the drawing plane and is

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not seen in the drawing), and the lower pendulum swings about an axis passing through the point  $M_1$  perpendicular to plane  $xOy$ .  $Ox$  axis has a horizontal direction, whereas  $Oy$  axis is directed downwards normal to that (Fig. 1).

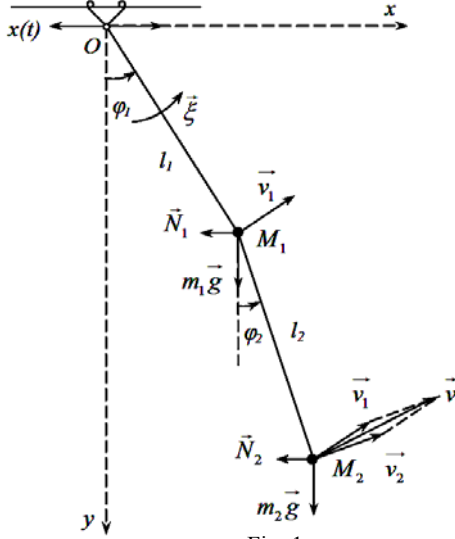


Fig. 1.

If we assume, that the upper suspension center  $O$  moves in the horizontal direction according to the given law, then the gravity forces  $m_1g$ ,  $m_2g$  and translational inertia loads  $N_1 = m_1\ddot{x}$  and  $N_2 = m_2\ddot{x}$  will act on the material particles  $M_1$  and  $M_2$  respectively.

Assume also, that the device at the suspension center acts on  $OM_1$  rod of pendulum with controlling moment  $\xi$ .

The coordinates of system will be uniquely determined by angles  $\varphi_1$ ,  $\varphi_2$  made by the vertical and rods  $l_1$ ,  $l_2$  respectively. The equations of motion of

the system are derived from the Lagrange equations [8]

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\varphi}_i} \right) - \frac{\partial T}{\partial \varphi_i} = Q_i - \frac{\partial \Pi}{\partial \varphi_i}, \quad i=1,2, \quad (1)$$

where  $T$  and  $\Pi$  are the kinetic and potential energies respectively, and  $Q_i$  are generalized forces. The velocity of point  $M_1$   $\vec{v}_1$  is equal to  $v_1 = l_1\dot{\varphi}_1$ , and the square of the velocity of point  $M_2$  is written as

$$v^2 = v_1^2 + v_2^2 + 2v_1v_2 \cos \alpha = l_1^2\dot{\varphi}_1^2 + l_2^2\dot{\varphi}_2^2 + 2l_1l_2\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2).$$

The kinetic and potential energies can be written as:

$$T = \frac{m_1 l_1^2}{2} \dot{\varphi}_1^2 + \frac{m_2}{2} (l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)), \quad (2)$$

$$\Pi = m_1 g l_1 (1 - \cos \varphi_1) + m_2 g (l_1 (1 - \cos \varphi_1) + l_2 (1 - \cos \varphi_2)). \quad (3)$$

The generalized forces  $Q_i$  effecting the system are expressed as:

$$Q_1 = -m_1 l_1 \ddot{x} \cos \varphi_1 - m_2 (l_1 \cos \varphi_1 + l_2 \cos \varphi_2) \ddot{x} + \xi = -((m_1 + m_2) l_1 \cos \varphi_1 + m_2 l_2 \cos \varphi_2) \ddot{x} + \xi, \quad (4)$$

$$Q_2 = -m_2 l_2 \ddot{x} \cos \varphi_2. \quad (5)$$

If we insert expressions (2)–(5) into equation (1), we get

$$\begin{cases} (m_1 + m_2) l_1^2 \ddot{\varphi}_1 + m_2 l_1 l_2 \ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + m_2 l_1 l_2 \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) = \\ = -(m_1 + m_2) g l_1 \sin \varphi_1 - ((m_1 + m_2) l_1 \cos \varphi_1 + m_2 l_2 \cos \varphi_2) \ddot{x} + \xi, \\ m_2 l_2^2 \ddot{\varphi}_2 + m_2 l_1 l_2 \ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) - m_2 l_1 l_2 \dot{\varphi}_1^2 \sin(\varphi_1 - \varphi_2) = -m_2 g l_2 \sin \varphi_2 - m_2 l_2 \ddot{x} \cos \varphi_2. \end{cases} \quad (6)$$

The system (6) may be linearized by assuming, that angles  $\varphi_1$  and  $\varphi_2$  are small, i.e.  $\cos(\varphi_1 - \varphi_2) \approx 1$ ,  $\sin(\varphi_1 - \varphi_2) \approx \varphi_1 - \varphi_2$ ,  $\sin \varphi_1 \approx \varphi_1$ ,  $\sin \varphi_2 \approx \varphi_2$ . We have

$$\begin{cases} (m_1 + m_2) l_1^2 \ddot{\varphi}_1 + m_2 l_1 l_2 \ddot{\varphi}_2 = -(m_1 + m_2) g l_1 \varphi_1 - ((m_1 + m_2) l_1 + m_2 l_2) \ddot{x} + \xi, \\ m_2 l_2^2 \ddot{\varphi}_2 + m_2 l_1 l_2 \ddot{\varphi}_1 = -m_2 g l_2 \varphi_2 - m_2 l_2 \ddot{x}; \end{cases} \quad (7)$$

$$\begin{cases} \ddot{\varphi}_1 = -\frac{m_1+m_2}{m_1} \cdot \frac{g}{l_1} \varphi_1 + \frac{m_2}{m_1} \cdot \frac{g}{l_1} \varphi_2 - \frac{m_1 l_1 + m_2 l_2}{m_1 l_1^2} \ddot{x} + \frac{1}{m_1 l_1^2} \xi, \\ \ddot{\varphi}_2 = \frac{m_1+m_2}{m_1} \cdot \frac{g}{l_2} \varphi_1 - \frac{m_1+m_2}{m_1} \cdot \frac{g}{l_2} \varphi_2 + \frac{m_2}{m_1 l_1} \ddot{x} - \frac{1}{m_1 l_1 l_2} \xi. \end{cases} \quad (8)$$

Now make the following notations:  $D = -\frac{m_1+m_2}{m_1} \cdot \frac{g}{l_1} < 0$ ,  $H = \frac{m_2}{m_1} \cdot \frac{g}{l_1} > 0$ ,  $R = \frac{m_1+m_2}{m_1} \cdot \frac{g}{l_2} > 0$ ,  $S = -\frac{m_1+m_2}{m_1} \cdot \frac{g}{l_2} < 0$ ,  $P = -\frac{m_1 l_1 + m_2 l_2}{m_1 l_1^2}$ ,  $W = \frac{m_2}{m_1 l_1}$ . If we introduce these notations in the set (8), we get

$$\begin{cases} \ddot{\varphi}_1 = D\varphi_1 + H\varphi_2 + P\ddot{x} + \frac{1}{m_1 l_1^2} \xi, \\ \ddot{\varphi}_2 = R\varphi_1 + S\varphi_2 + W\ddot{x} - \frac{1}{m_1 l_1 l_2} \xi. \end{cases} \quad (9)$$

Let us introduce dimensionless quantities by the following notation:  $\tau = kt$ , where  $k$  has the dimension of frequency. Hence,

$$\begin{cases} \frac{d^2 \varphi_1}{d\tau^2} = \frac{D}{k^2} \varphi_1 + \frac{H}{k^2} \varphi_2 + P \frac{d^2 f(\tau)}{d\tau^2} + \frac{1}{k^2 m_1 l_1^2} \xi, \\ \frac{d^2 \varphi_2}{d\tau^2} = \frac{R}{k^2} \varphi_1 + \frac{S}{k^2} \varphi_2 + W \frac{d^2 f(\tau)}{d\tau^2} - \frac{1}{k^2 m_1 l_1 l_2} \xi, \end{cases} \quad (10)$$

where  $f(\tau) = x\left(\frac{1}{k}\tau\right)$ . (11)

Now make the following notations:  $d = \frac{D}{k^2}$ ,  $h = \frac{H}{k^2}$ ,  $r = \frac{R}{k^2}$ ,  $s = \frac{S}{k^2}$ ,  $u = \frac{1}{k^2 m_1 l_1^2} \xi$ ,  $L = \frac{l_1}{l_2}$ . Thus, the system (10) can be written as:

$$\begin{cases} \frac{d^2 \varphi_1}{d\tau^2} = d\varphi_1 + h\varphi_2 + P \frac{d^2 f(\tau)}{d\tau^2} + u, \\ \frac{d^2 \varphi_2}{d\tau^2} = r\varphi_1 + s\varphi_2 + W \frac{d^2 f(\tau)}{d\tau^2} - Lu. \end{cases} \quad (12)$$

If we introduce in system (12) the phase coordinates by means of the following notations:  $x_1 = \varphi_1$ ,  $x_2 = \frac{d\varphi_1}{d\tau}$ ,  $x_3 = \varphi_2$ ,  $x_4 = \frac{d\varphi_2}{d\tau}$ , then

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = dx_1 + hx_3 + P \frac{d^2 f(\tau)}{d\tau^2} + u, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = rx_1 + sx_3 + W \frac{d^2 f(\tau)}{d\tau^2} - Lu. \end{cases} \quad (13)$$

Here for simplicity the notation  $\dot{x}_i = \frac{dx_i}{d\tau}$  ( $i=1,2,3,4$ ) was adopted. The obtained system (13) is a controlled system of linear nonhomogeneous differential equations.

For the existence and uniqueness of the solution of system (13) it is necessary and sufficient that the system be completely controlled, i.e.  $\text{rank}K = 4$ .  $K$  is the Kalman matrix of the form

$$K = \begin{pmatrix} 0 & 1 & 0 & d - Lh \\ 1 & 0 & d - Lh & 0 \\ 0 & -L & 0 & r - Ls \\ -L & 0 & r - Ls & 0 \end{pmatrix}.$$

It is obvious, that  $\text{rank}K = 4$ . Hence, the solution  $u^0$  of system (13) exists and is unique [9].

Now, let us formulate the optimum stabilization problem for the system (13).

**Problem.** It is required to find an optimal control action  $u^0$  such that it makes the solution  $x_1 = x_2 = x_3 = x_4 = 0$  of system (13) asymptotically stable and satisfies the constraint

$$\mathcal{J}[u] = \int_0^{\infty} (x_2^2 + x_4^2 + u^2) d\tau \rightarrow \min. \quad (14)$$

**Solution.** For the solution of the problem an assumption is made that the double pendulum is composed of two pendulums of equal masses  $m$  with equal lengths  $l$ , and the suspension point is moving, according to  $x = ne^{-\alpha t}$  ( $n \neq 0$ ) law in the horizontal direction.

In this case, the equations of motion (13) become

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2x_1 + x_3 - 2pe^{-\alpha\tau/k} + u, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = 2x_1 - 2x_3 + pe^{-\alpha\tau/k} - u. \end{cases} \quad (15)$$

Here we have  $k = \sqrt{\frac{g}{l}}$ ,  $p = \frac{n\alpha^2}{gk^2} = \frac{n\alpha^2}{g^2}$ ,  $u = \frac{1}{ml^2k^2}\xi$ , and for simplicity

we make the following notation  $\dot{x}_i = \frac{dx_i}{d\tau}$  ( $i=1,2,3,4$ ). To solve the optimal stabilization problem we use the Lyapunov–Bellman method [9]. In this case the Bellman expression can be written as

$$B[\cdot] = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 + \frac{\partial V}{\partial x_3} \dot{x}_3 + \frac{\partial V}{\partial x_4} \dot{x}_4 + x_2^2 + x_4^2 + u^2, \quad (16)$$

where  $V = V(x_1, x_2, x_3, x_4, \tau)$  is the Lyapunov function for system (15). Now, substitute expressions for  $\dot{x}_1$ ,  $\dot{x}_2$ ,  $\dot{x}_3$  and  $\dot{x}_4$  from (15) into (16). We get

$$B[\cdot] = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial x_1} x_2 + \frac{\partial V}{\partial x_2} \left( -2x_1 + x_3 - 2pe^{-\frac{\alpha}{k}\tau} + u \right) + \frac{\partial V}{\partial x_3} x_4 + \frac{\partial V}{\partial x_4} \left( 2x_1 - 2x_3 + pe^{-\frac{\alpha}{k}\tau} - u \right) + x_2^2 + x_4^2 + u^2. \quad (17)$$

As  $\frac{\partial B[\cdot]}{\partial u} \Big|_{u=u^0} = 0$ , we have  $\frac{\partial B[\cdot]}{\partial u} \Big|_{u=u^0} = \frac{\partial V}{\partial x_2} - \frac{\partial V}{\partial x_4} + 2u^0 = 0$ . Thus, we obtain

$$u^0 = -\frac{1}{2} \left( \frac{\partial V}{\partial x_2} - \frac{\partial V}{\partial x_4} \right). \quad (18)$$

As in the case of optimal control  $u = u^0$ , the Bellman expression assumes the value of 0, we can substitute (18) in (17) and require it to be equal 0. Hence, we obtain

$$B[\cdot] \Big|_{u=u^0} = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial x_1} x_2 + \frac{\partial V}{\partial x_2} \left( -2x_1 + x_3 - 2pe^{-\frac{\alpha}{k}\tau} \right) + \frac{\partial V}{\partial x_3} x_4 + \frac{\partial V}{\partial x_4} \left( 2x_1 - 2x_3 + pe^{-\frac{\alpha}{k}\tau} \right) + x_2^2 + x_4^2 - \frac{1}{4} \left( \frac{\partial V}{\partial x_2} - \frac{\partial V}{\partial x_4} \right)^2 = 0. \quad (19)$$

The Lyapunov function is sought for in the following form [9]:

$$V(x_1, x_2, x_3, x_4, \tau) = V^{(2)}(x_1, x_2, x_3, x_4) + V^{(1)}(x_1, x_2, x_3, x_4, \tau) + V^{(0)}(\tau), \quad (20)$$

where the function  $V^{(2)}(x_1, x_2, x_3, x_4)$  is a quadratic form depending on variables  $x_1, x_2, x_3, x_4$ , the  $V^{(1)}(x_1, x_2, x_3, x_4, \tau)$  is a linear function of variables  $x_1, x_2, x_3, x_4$  with variable coefficients, and  $V^{(0)}(\tau)$  is a continuous function of only time  $\tau$ .

Now introduce the expression of Lyapunov function (20) into (19). We have

$$\begin{aligned} & \frac{\partial(V^{(2)} + V^{(1)} + V^{(0)})}{\partial \tau} + \frac{\partial(V^{(2)} + V^{(1)} + V^{(0)})}{\partial x_1} x_2 + \frac{\partial(V^{(2)} + V^{(1)} + V^{(0)})}{\partial x_2} \left( -2x_1 + x_3 - 2pe^{-\frac{\alpha}{k}\tau} \right) + \\ & + \frac{\partial(V^{(2)} + V^{(1)} + V^{(0)})}{\partial x_3} x_4 + \frac{\partial(V^{(2)} + V^{(1)} + V^{(0)})}{\partial x_4} \left( 2x_1 - 2x_3 + pe^{-\frac{\alpha}{k}\tau} \right) + x_2^2 + x_4^2 - \\ & - \frac{1}{4} \left( \frac{\partial(V^{(2)} + V^{(1)} + V^{(0)})}{\partial x_2} - \frac{\partial(V^{(2)} + V^{(1)} + V^{(0)})}{\partial x_4} \right)^2 = 0. \end{aligned} \quad (21)$$

Now separate the second order terms depending on variables  $x_1, x_2, x_3, x_4$ , the linear terms depending on variables  $x_1, x_2, x_3, x_4$  and the terms independent of variables  $x_1, x_2, x_3, x_4$  in equation (21) and equate them to 0 separately. For the second order terms depending on variables  $x_1, x_2, x_3, x_4$ , we have

$$\frac{\partial V^{(2)}}{\partial x_1} x_2 + \frac{\partial V^{(2)}}{\partial x_2} (-2x_1 + x_3) + \frac{\partial V^{(2)}}{\partial x_3} x_4 + \frac{\partial V^{(2)}}{\partial x_4} (2x_1 - 2x_3) + x_2^2 + x_4^2 - \frac{1}{4} \left( \frac{\partial V^{(2)}}{\partial x_2} - \frac{\partial V^{(2)}}{\partial x_4} \right)^2 = 0 \quad (22)$$

for linear terms depending on variables  $x_1, x_2, x_3, x_4$ , we have

$$\begin{aligned} & \frac{\partial V^{(1)}}{\partial \tau} + \frac{\partial V^{(1)}}{\partial x_1} x_2 - 2pe^{-\frac{\alpha}{k}\tau} \frac{\partial V^{(2)}}{\partial x_2} + \frac{\partial V^{(1)}}{\partial x_2} (-2x_1 + x_3) + \frac{\partial V^{(1)}}{\partial x_3} x_4 + pe^{-\frac{\alpha}{k}\tau} \frac{\partial V^{(2)}}{\partial x_4} + \\ & + \frac{\partial V^{(1)}}{\partial x_4} (2x_1 - 2x_3) - \frac{1}{2} \left( \frac{\partial V^{(2)}}{\partial x_2} - \frac{\partial V^{(2)}}{\partial x_4} \right) \left( \frac{\partial V^{(1)}}{\partial x_2} - \frac{\partial V^{(1)}}{\partial x_4} \right) = 0. \end{aligned} \quad (23)$$

And for terms depending only on time  $\tau$ , we have

$$\frac{dV^{(0)}}{d\tau} - 2pe^{-\frac{\alpha}{k}\tau} \frac{\partial V^{(1)}}{\partial x_2} + pe^{-\frac{\alpha}{k}\tau} \frac{\partial V^{(1)}}{\partial x_4} - \frac{1}{4} \left( \frac{\partial V^{(1)}}{\partial x_2} - \frac{\partial V^{(1)}}{\partial x_4} \right)^2 = 0. \quad (24)$$

By solving the obtained system of equations (22)–(24), we find  $V^{(2)}(x_1, x_2, x_3, x_4)$ ,  $V^{(1)}(x_1, x_2, x_3, x_4, \tau)$  and  $V^{(0)}(\tau)$  functions.

As, in the case of  $f(\tau) = 0$ , the system (13) is completely controlled, the function  $V^{(2)}(x_1, x_2, x_3, x_4)$  can be uniquely determined. We seek the function in the following form:

$$\begin{aligned} V^{(2)}(x_1, x_2, x_3, x_4) = & \frac{1}{2} (c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + c_{44}x_4^2 + 2c_{12}x_1x_2 + 2c_{13}x_1x_3 + \\ & + 2c_{14}x_1x_4 + 2c_{23}x_2x_3 + 2c_{24}x_2x_4 + 2c_{34}x_3x_4). \end{aligned} \quad (25)$$

For determination of coefficients  $c_{ij}$  ( $i, j=1, 2, 3, 4$ ), we have the following system of algebraic equations:

$$\left\{ \begin{aligned} -2c_{12} + 2c_{14} - \frac{1}{4}(c_{12} - c_{14})^2 &= 0, \\ 1 + c_{12} - \frac{1}{4}(c_{22} - c_{24})^2 &= 0, \\ c_{23} - 2c_{34} - \frac{1}{4}(c_{23} - c_{34})^2 &= 0, \\ 1 + c_{34} - \frac{1}{4}(c_{24} - c_{44})^2 &= 0, \\ c_{11} - 2c_{22} + 2c_{24} - \frac{1}{2}(c_{12} - c_{14})(c_{22} - c_{24}) &= 0, \\ c_{12} - 2c_{14} - 2c_{23} + 2c_{34} - \frac{1}{2}(c_{12} - c_{14})(c_{23} - c_{34}) &= 0, \\ c_{13} - 2c_{24} + 2c_{44} - \frac{1}{2}(c_{12} - c_{14})(c_{24} - c_{44}) &= 0, \\ c_{13} + c_{22} - 2c_{24} - \frac{1}{2}(c_{22} - c_{24})(c_{23} - c_{34}) &= 0, \\ c_{14} + c_{23} - \frac{1}{2}(c_{22} - c_{24})(c_{24} - c_{44}) &= 0, \\ c_{24} + c_{33} - 2c_{44} - \frac{1}{2}(c_{23} - c_{34})(c_{24} - c_{44}) &= 0. \end{aligned} \right. \quad (26)$$

From solutions of system (26) we choose that one, in case of which, the function  $V^{(2)}(x_1, x_2, x_3, x_4)$  is positively defined. For our system the solution has the following form:

$$\begin{aligned} c_{11}^0 &\approx 6.515172, \quad c_{12}^0 \approx 1.652967, \quad c_{13}^0 \approx -0.209687, \quad c_{14}^0 \approx 1.652967, \\ c_{22}^0 &\approx 7.651655, \quad c_{23}^0 \approx -1.823735, \quad c_{24}^0 \approx 4.394069, \quad c_{33}^0 \approx 4.647082, \\ c_{34}^0 &\approx -0.997252, \quad c_{44}^0 \approx 4.498912 \end{aligned} \quad (27)$$

(these coefficients were determined by means of computer program Mathematica 7 to within  $10^{-6}$  accuracy).

Now insert these coefficients in expression (25). We obtain

$$\begin{aligned} V^{(2)}(x_1, x_2, x_3, x_4) &= \frac{1}{2}(c_{11}^0 x_1^2 + c_{22}^0 x_2^2 + c_{33}^0 x_3^2 + c_{44}^0 x_4^2 + 2c_{12}^0 x_1 x_2 + 2c_{13}^0 x_1 x_3 + \\ &+ 2c_{14}^0 x_1 x_4 + 2c_{23}^0 x_2 x_3 + 2c_{24}^0 x_2 x_4 + 2c_{34}^0 x_3 x_4) = 3.257586x_1^2 + 3.825827x_2^2 + \\ &+ 2.323541x_3^2 + 2.249456x_4^2 + 1.652967x_1 x_2 - 0.209687x_1 x_3 + 1.652967x_1 x_4 - \\ &- 1.823735x_2 x_3 + 4.394069x_2 x_4 - 0.997252x_3 x_4. \end{aligned} \quad (28)$$

Now determine  $V^{(1)}(x_1, x_2, x_3, x_4, \tau)$  function from equation (23). If (28) is introduced in (23), we have

$$\begin{aligned} \frac{\partial V^{(1)}}{\partial \tau} + \frac{\partial V^{(1)}}{\partial x_1} x_2 + \frac{\partial V^{(1)}}{\partial x_2} (-2x_1 + x_3) + \frac{\partial V^{(1)}}{\partial x_3} x_4 + \frac{\partial V^{(1)}}{\partial x_4} (2x_1 - 2x_3) + \\ + pe^{\frac{\alpha}{k}\tau} (-2(c_{22}^0 x_2 + c_{12}^0 x_1 + c_{23}^0 x_3 + c_{24}^0 x_4) + (c_{44}^0 x_4 + c_{14}^0 x_1 + c_{24}^0 x_2 + c_{34}^0 x_3)) - \\ - \frac{1}{2}((c_{22}^0 x_2 + c_{12}^0 x_1 + c_{23}^0 x_3 + c_{24}^0 x_4) - (c_{44}^0 x_4 + c_{14}^0 x_1 + c_{24}^0 x_2 + c_{34}^0 x_3)) \left( \frac{\partial V^{(1)}}{\partial x_2} - \frac{\partial V^{(1)}}{\partial x_4} \right) = 0. \end{aligned} \quad (29)$$

The function  $V^{(1)}(x_1, x_2, x_3, x_4, \tau)$  is sought in the following form:

$$V^{(1)}(x_1, x_2, x_3, x_4, \tau) = y_1(\tau)x_1 + y_2(\tau)x_2 + y_3(\tau)x_3 + y_4(\tau)x_4, \quad (30)$$

where  $y_1(\tau)$ ,  $y_2(\tau)$ ,  $y_3(\tau)$  and  $y_4(\tau)$  are unknown functions of  $\tau$ . Inserting (30) in (29), we have

$$\begin{aligned} \dot{y}_1 x_1 + \dot{y}_2 x_2 + \dot{y}_3 x_3 + \dot{y}_4 x_4 + y_1 x_2 + y_2 (-2x_1 + x_3) + y_3 x_4 + y_4 (2x_1 - 2x_3) + \\ + pe^{\frac{\alpha}{k}\tau} ((-2c_{12}^0 + c_{14}^0)x_1 + (-2c_{22}^0 + c_{24}^0)x_2 + (-2c_{23}^0 + c_{34}^0)x_3 + (-2c_{24}^0 + c_{44}^0)x_4) - \\ - \frac{1}{2}(y_2 - y_4)((c_{12}^0 - c_{14}^0)x_1 + (c_{22}^0 - c_{24}^0)x_2 + (c_{23}^0 - c_{34}^0)x_3 + (c_{24}^0 - c_{44}^0)x_4) = 0. \end{aligned} \quad (31)$$

Now separate the coefficients of variables  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$  and equate them to 0 separately. Thus, we obtain a system of linear nonhomogeneous differential equations for determination of  $y_1(\tau)$ ,  $y_2(\tau)$ ,  $y_3(\tau)$  and  $y_4(\tau)$  functions:

$$\begin{cases} \dot{y}_1 = a_1 y_2 - a_1 y_4 - pb_1 e^{\frac{\alpha}{k}\tau}, \\ \dot{y}_2 = -y_1 + a_2 y_2 - a_2 y_4 - pb_2 e^{\frac{\alpha}{k}\tau}, \\ \dot{y}_3 = a_3 y_2 - (a_3 - 1)y_4 - pb_3 e^{\frac{\alpha}{k}\tau}, \\ \dot{y}_4 = a_4 y_2 - y_3 - a_4 y_4 - pb_4 e^{\frac{\alpha}{k}\tau}, \end{cases} \quad (32)$$

where the following notations were made:

$$2 + \frac{c_{12}^0 - c_{14}^0}{2} = a_1, \quad \frac{c_{22}^0 - c_{24}^0}{2} = a_2, \quad -1 + \frac{c_{23}^0 - c_{34}^0}{2} = a_3, \quad \frac{c_{24}^0 - c_{44}^0}{2} = a_4, \\ -2c_{12}^0 + c_{14}^0 = b_1, \quad -2c_{22}^0 + c_{24}^0 = b_2, \quad -2c_{23}^0 + c_{34}^0 = b_3, \quad -2c_{24}^0 + c_{44}^0 = b_4.$$

We have for numerical values of coefficients

$$a_1 \approx 2, \quad a_2 \approx 1.628793, \quad a_3 \approx -1.413242, \quad a_4 \approx -0.052422, \\ b_1 \approx -1.65297, \quad b_2 \approx -10.9092, \quad b_3 \approx 2.65022, \quad b_4 \approx -4.28923. \quad (33)$$

Here for simplicity we adopted the following notations  $\dot{x}_i = \frac{dx_i}{d\tau}$  ( $i=1,2,3,4$ ). The solution of system (32) is determined by using the Cauchy formula [10]:

$$y_1^o(\tau) = kpe^{-\frac{\alpha}{k}\tau} \left( \frac{-1.900079 + 3.951708k}{1 + 0.228166k + 0.569432k^2} + \frac{0.247112 + 9.213856k}{1 + 1.45304k + 3.51227k^2} \right), \quad (34)$$

$$y_2^o(\tau) = kpe^{-\frac{\alpha}{k}\tau} \left( \frac{-8.487157 - 0.265233k}{1 + 0.228166k + 0.569432k^2} + \frac{-2.422084 + 3.939016k}{1 + 1.45304k + 3.51227k^2} \right), \quad (35)$$

$$y_3^o(\tau) = kpe^{-\frac{\alpha}{k}\tau} \left( \frac{1.826889 + 3.612668k}{1 + 0.228166k + 0.569432k^2} + \frac{0.82333 - 7.065952k}{1 + 1.45304k + 3.51227k^2} \right), \quad (36)$$

$$y_4^o(\tau) = kpe^{-\frac{\alpha}{k}\tau} \left( \frac{-6.294537 + 0.275749k}{1 + 0.228166k + 0.569432k^2} + \frac{-2.005312 + 0.591238k}{1 + 1.45304k + 3.51227k^2} \right). \quad (37)$$

The optimal function  $V^{(1)}(x_1, x_2, x_3, x_4, \tau)$  is determined by inserting  $y_1(\tau)$ ,  $y_2(\tau)$ ,  $y_3(\tau)$  and  $y_4(\tau)$  functions from (34), (35), (36), (37) in (30).

As for the optimal control action we have expression (18) and the function  $V^{(0)}$  is only  $\tau$  dependent, i.e.  $\frac{\partial V^{(0)}}{\partial x_2} = \frac{\partial V^{(0)}}{\partial x_4} \equiv 0$ , then we may omit the calculation

of  $V^{(0)}$  function. Consequently, the optimal control action can be determined by means of the following expression:

$$u^0 = -\frac{1}{2} \left( \frac{\partial V^{(2)}}{\partial x_2} - \frac{\partial V^{(2)}}{\partial x_4} + \frac{\partial V^{(1)}}{\partial x_2} - \frac{\partial V^{(1)}}{\partial x_4} \right). \quad (38)$$

Now insert (25) and (30) into (38). We obtain

$$u^0 = -\frac{1}{2} ((c_{12}^0 - c_{14}^0)x_1 + (c_{22}^0 - c_{24}^0)x_2 + (c_{23}^0 - c_{34}^0)x_3 + (c_{24}^0 - c_{44}^0)x_4) - \\ -\frac{1}{2} y_2^0(\tau) + \frac{1}{2} y_4^0(\tau). \quad (39)$$

If we insert (27), (35) and (37) in (40) and do simplifications, then we obtain for  $u^0$

$$u^0(x_1, x_2, x_3, x_4, \tau) = -1.62879x_2 + 0.413242x_3 + 0.0524218x_4 - \\ -kpe^{-\frac{\alpha}{k}\tau} \frac{(-1.990251 + 0.001563k)(0.327772 + 0.059833k + k^2)}{(1.756139 + 0.40069k + k^2)(0.284716 + 0.413706k + k^2)}. \quad (41)$$

Thus, the optimal control action  $u^0$  determined by means of expression (41) is the required control action.



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**ՄԵԽԱՆԻԿԱ****Ա Մ Փ Ո Փ ՈՒ Մ**

**Ս. Գ. Շահինյան, Գ. Ն. Կիրակոսյան.** Շարժական կախման կետ ունեցող կրկնակի մաթեմատիկական ճոճանակի օպտիմալ կայունացման մասին էջ. 31–39

Աշխատանքում դիտարկվում է շարժական կախման կետ ունեցող կրկնակի մաթեմատիկական ճոճանակի օպտիմալ կայունացման խնդիրը, երբ առաջին ձողի կախման կետը տրված օրենքով շարժվում է հորիզոնական ուղղությամբ: Խնդիրը բերված է գծային անհամասեռ համակարգի օպտիմալ կայունացման խնդրին, որը լուծված է հավասար երկարություններ և հավասար զանգվածներ ունեցող ճոճանակի համար: Կառուցված է Լյապունովի օպտիմալ ֆունկցիան և ստացված է օպտիմալ ղեկավարող ազդեցությունը:

**С. Г. Шагинян, Г. Н. Киракосян.** Об оптимальной стабилизации двойного математического маятника с подвижной точкой подвеса стр. 31–39

**А Н Н О Т А Ц И Я**

В работе рассматривается задача оптимальной стабилизации двойного математического маятника, когда точка подвеса движется по заданному закону в горизонтальном направлении. Задача приведена к задаче оптимальной стабилизации линейной неоднородной системы, которая решена для случая, когда у первого и второго маятника одинаковые массы и длины. Построена оптимальная функция Ляпунова и оптимальное управляющее воздействие.