About Stability of Dynamical Systems With Integrally Small Perturbations

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Abstract. The presented paper deals with the problems of stability of motion of dynamical systems when integrally small perturbations forces affect the system in the finite time interval. It is assumed that the perturbation force is selected from a rather broad class of functions, of a class of generalized functions (they may be impulse functions). The need for such studies is dictated by the practice, when you have to take into account the continuous perturbation or impulse effects that can qualitatively change the behavior of the object. Feature of research is that we are not going to follow the "letter" of the classical notion of stability but propose a modification of the concept, tailored applications. So, we admit the possibility of strong perturbations of the object at a finite time interval. If after a specified period of time perturbations in the motion of the object will be small and will remain little later, then these movements is called stable under small perturbations of the integral. Obviously, such notion of stability is different from the Lyapunov stability, since it permits "fluctuation" instability at a certain interval. However, this instability is not fatal to the nature of the object, since replaced by the steady state of motion.

INTRODUCTION

The problem of stability of dynamic systems with different perturbation factors was studied by many authors. One of the first who attached great theoretical and practical importance to the theory of stability with perturbation factors was N.G.Chetaev [1]. Later appeared the works by I.G. Malkin [2], N.N. Krasovsky [3], J.L.Massera [4] and others. In the work [1] the influence of small perturbing forces on the stability of motion of a dynamical system is investigated. In [2] the theorem on stability under constantly acting small perturbations is proved. In [3] the sufficient conditions are determined under which the problem of stability with constantly acting perturbations, bounded on average, is solved. In [5,6] the influence of the force structure on the stability of motion is studied. In [5] the conditions of stability under parametric perturbations are obtained. The problem of stability with impulse action is investigated in [7]-[10].

1. MAIN CONCEPTS AND DEFINITIONS

A. Problem Set-up

Let's consider the system of differential equations

$$\dot{x}_i = F_i(t, x_1, \ldots, x_n) \qquad i = 1,2,\ldots,n, \tag{1}$$

in which $F_i(t, x_1, ..., x_n): R^{n+1} \to R^1$ are continuous functions satisfying all the conditions for solving the system of equations existing and being unique in the domain

$$x_i \in R^1; \quad t \in [t_0; +\infty) \tag{2}$$

besides, $F_i(t, 0, 0, ..., 0) \equiv 0; \quad i = 1, 2, ..., n.$

Let perturbation forces $\varphi_i(t)$ act on system (1) satisfying the condition

$$\text{a)} \left\| \int_{t_0}^{T} \varphi(t) dt \right\| = \left[\sum_{i=1}^{n} \left(\int_{t_0}^{T} \varphi_i(t) dt \right)^2 \right]^{1/2} < \infty, \tag{3a}$$

$$\text{b)} \ \varphi_i(t) \equiv 0, \text{ when } t \geq T, i = 1, ..., n. \tag{3b}$$

Here $T > t_0$ is the given magnitude.

Differential equations of system (1) having perturbation forces $\varphi_i(t)$ will be reduced to the form

$$\dot{x}_i = F_i(t, x_1, ..., x_n) + \varphi_i(t), \quad (i = 1, 2, ..., n). \tag{4}$$

Let's assume that $\varphi_i(t)$ perturbations are such that the solution of systems (4) also exists for every $t \geq t_0$. Let's denote a certain class of $\varphi(t)$ by Φ which will satisfy conditions (3a), (3b).

Let's formulate the following problems:

Problem 1. Define the conditions when for every $\varepsilon > 0$ there exists number $\delta > 0$ and the moment of time $t_* \geq T$, so that $\|x(t)\| < \varepsilon$ for all $t \geq t_*$ if $\|x(t_0)\| < \delta$ and $\left\| \int_{t_0}^{T} \varphi(t) dt \right\| < \delta$ where $x(t)$ is the solution of system (4), and $\varphi(t)$ is any perturbation from class Φ.

Problem 2. Define the conditions when for every $\varepsilon > 0$ there exists number $\delta > 0$ and the moment of time $t_* \geq T$, so that $\|x(t)\| < \varepsilon$ for all $t \geq t_*$ and $\lim_{t \to \infty} \|x(t)\| = 0$, if $\|x(t_0)\| < \delta$ and $\left\| \int_{t_0}^{T} \varphi(t) dt \right\| < \delta$ where $x(t)$ is the solution of system (4), and $\varphi(t)$ is any perturbation from class Φ.

B. Definitions of Stability, Non-stability and Asymptotic Stability with Integrally Small Perturbations

In point A the problem set-up of stability is given when perturbation forces $\varphi_i(t)$, which satisfy the conditions in (3a), (3b), act at the final interval of time $[t_0, T]$, that is, perturbation forces $\varphi_i(t)$ are chosen from quite a wide class of functions, from the class of generalized functions (integrally small perturbations).

There is no purpose in studying such problems in the frame of Lyapunov's classical theory, as within the intervals where the perturbation forces act the solutions of the systems can be discontinuous functions. Therefore, it is necessary to give new definitions of stability: stability with integrally small perturbations and study the problems set up from this point of view.

Let's bring the following definitions:

Definition 1. Solution $x = 0$ of system (1) is said to be stable with integrally small perturbations if for every number $\varepsilon > 0$ there exists such number $\delta > 0$ and moment of time $t_* \geq T$ that for every solution $x(t)$ of system (4)

$$\|x(t)\| < \varepsilon \text{ for every } t \geq t_*, \text{ if } \|x(t_0)\| < \delta \text{ and } \left\| \int_{t_0}^{T} \varphi(t) dt \right\| < \delta, \tag{5}$$

in which every $\varphi(t)$ perturbation is from class Φ (Fig.1).

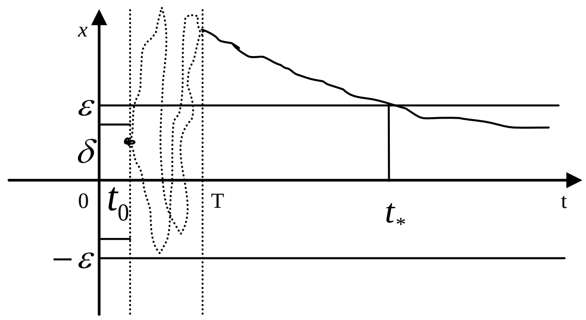

FIGURE 1.

Otherwise, solution $x = 0$ of system (1) is called non-stable with integrally small perturbations.

For example, solution $x = 0$ of system (1) will be non-stable with integrally small perturbations if for certain $\varepsilon > 0$ and every $\delta > 0$ there exists force $\varphi(t)$ which satisfies conditions (3a), (3b), solution $x = x(t)$ of system (4) (at least one) are such that $\|x(t_0)\| < \delta$, $\left\|\int_{t_0}^{T} \varphi(t)dt\right\| < \delta$, but $\|x(t)\| \geq \varepsilon$ for every $t \geq T$.

Definition 2. Solution $x = 0$ of system (1) is said to be asymptotically stable with integrally small perturbations if it is stable with integrally small perturbations and $\lim_{t\to\infty} \|x(t)\| = 0$.

(Here $x(t)$ is the solution of system (4)).

2. THE QUESTIONS OF STABILITY WITH INTEGRALLY SMALL PERTURBATIONS FOR THE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Let's consider the system of linear differential equations with constant coefficients

$$\dot{x} = Ax, \tag{6}$$

where $x \in R^n$; A is the $n \times n$ constant matrix.

Let's also consider the system of differential equations

$$\dot{x} = Ax + \varphi(t) \tag{7}$$

in which vector-function $\varphi(t): R^1 \to R^n$ satisfies all the conditions of point A.

First problems 1 and 2 are solved for system (6) by Lyapunov's method.

The roots of the characteristic equation

$$|A - \lambda E| = 0 \tag{8}$$

are denoted by $\lambda_1, \lambda_2, ..., \lambda_l$ with multiples $r_1, r_2, ..., r_l$, correspondingly $\sum_{i=1}^{l} r_i = n$, $E - n \times n$ is a unit matrix.

Now the following theorems will be proved:

Theorem 1. System (6) is stable with integrally small perturbations if and only if the roots of equation (8) satisfy the following conditions:

1) $\lambda_1 = 0$ (with multiple r_1), besides, this root corresponds to simple elementary divisors ($(rangA = n - r_1)$

2) $Re\lambda_i < 0$, when $i = 2, ..., l$.

Theorem 2. For system (6) to be asymptotically stable with integrally small perturbations it is necessary and sufficient that the roots of equation (8) satisfy the conditions

$$Re\lambda_i < 0 \quad (i = 1,...,l) \tag{9}$$

Corollary 1. If real parts of all roots of equation (8) are negative ($Re\lambda_i < 0 \ (i = 1,...,l)$), theorem 1 is true for all perturbations $\varphi(t)$ satisfying conditions (3a), (3b).

Corollary 2. If in theorem 1 $r_1 = n$, system (6) is stable with integrally small perturbations.

Theorem 3. For system (6) to be non-stable with integrally small perturbations it is necessary and sufficient that roots of the characteristic equation (8) satisfy the following conditions:

at least for one from $\lambda_k \ (1 \le k \le l) \quad Re\lambda_k > 0$, or

$\lambda_k = 0$ with multiple $r_k \ (1 < r_k \le n)$, the simple elementary divisors would not correspond, or

at least for one of λ_k would be of the form $Re\lambda_k = 0; \ Im\lambda_k \ne 0.$

As it is known, one of the more effective methods of the study of stability of the movement is Lyapunov's direct (or second) method. Now it will be found out how the property of stability with integrally small perturbations of system (6) can be characterized. The method of constructing Lyapunov's function satisfying stability with integrally small perturbations of systems (6) will be given.

The following statements are correct:

Theorem 4. System (6) is stable with integrally small perturbations if and only if there is a definite function $V(x)$ the derivative of which due to time is true for (6) is a definite function $W(x)$ with the negative sign of $V(x)$ and in the subspace $W(x) = 0$ system (6) can have only a constant solutions.

Theorem 5. For Lyapunov's stable system(6) to be non-stable with integrally small perturbations it is necessary and sufficient for the definitely positive function V to exist the time derivative of which is taken along the trajectories of system (6) and be function of fixed sign $W = \dot{V} \le 0$ having the opposite sign of V, and in subspace $W = 0$ of system (6) it would allow at least one "oscillating" solution of the form

$$x_j(t) = c_1 \cos\omega t + c_2 \sin\omega t + R(t) \quad (1 \le j \le n), \tag{10}$$

where $(c_1^2 + c_2^2)\omega \ne 0, \quad R(t)$ is a function of time of the exponential type.

3. QUESTIONS OF STABILITY WITH INTEGRALLY SMALL PERTURBATIONS FOR THE SYSTEMS OF NON-LINEAR DIFFERENTIAL EQUATIONS

Let's consider the system of differential equations

$$\dot{x}_i = F_i(x_1,...,x_n), \tag{11}$$

where $F_i(x_1,...,x_n): R^n \to R^1$ are continuous functions satisfying Lipshits's conditions in some restricted area $G \subset R^n$ and $F_i(0,...,0) = 0; \quad i = 1,...,n.$

As one of the most effective methods of studying the stability of the movement is considered to be Lyapunov's direct (or second) method, we'll try to figure out how the property of stability with integrally small perturbations of system (11) will be possible to characterize by means of Lyapunov's function.

Let system (11) have independent first integrals of the form

$$\sum_{j=1}^{n} a_{ij} x_j = c_i = const \quad (i = 1,...,k; \ 1 \le k \le n), \tag{12}$$

where, $A = \|a_{ij}\|$ is a $k \times n$ constant matrix ($rangA = k$). It'll be shown that in this case there exists such a non-singular matrix C , that by means of linear transformation $y = Cx$ system (11) can be reduced to the form

$$\dot{y}_i = 0, \qquad (i = 1,...,k) \tag{13}$$

$$\dot{y}_j = \Phi_j(c_1,...,c_k, y_{k+1},...,y_n), \qquad (j = k+1,...,n) \tag{14}$$

The system

$$\dot{x}_i = F_i(x_1,...,x_n) + \varphi_i(t), \tag{15}$$

in which the function $\varphi_i(t)$ satisfies all the conditions mentioned in point 1.1 after transformation $y = Cx$ will have the following form

$$\dot{y}_i = \psi_i(t), \qquad (i = 1,...,k) \tag{16}$$

$$\dot{y}_j = \Phi_j(c_1,...,c_k, y_{k+1},..., y_n) + \psi_j(t), \qquad (j = k+1,...,n), \tag{17}$$

where the vector $\psi(t) = (\psi_1(t),...,\psi_n(t))^* = C \cdot \varphi(t)$ and at $\left\| \int_{t_0}^{T} \varphi(t)dt \right\| < \delta$, $\left\| \int_{t_0}^{T} \psi(t)dt \right\| = \left\| \int_{t_0}^{T} C \cdot \varphi(t)dt \right\| < \delta_1$ as C is a constant matrix.

Now the next statements are correct:

Theorem 6. If system (11) can have independent first integrals of the form

$$\sum_{j=1}^{n} a_{ij} x_j = c_i = const \quad (i = 1,...,k;\ 1 \le k \le n),$$

and for system (14) there exists a definite-positive function $V(y_{k+1},...,y_n)$ satisfying the condition $\lim\limits_{\|y\|_{n-k} \to \infty} V(y) = \infty$, the derivative of which along the trajectories of system (14) satisfy the conditions

a) $\dot{V} = W < 0$ not in K,

b) $\dot{V} = W = 0$ on K,

along c_i uniformly, where $K \subset \{(y_{k+1,...,}y_n)\} = R^{n-k}$ the diversity of points does not contain whole trajectories of system (14) when $T \le t < \infty$, the trivial solution of system (11) is stable with integrally small perturbations.

Theorem 7. If non-linear system (11) differs from the trivial and makes definite-positive first integral, its trivial solution is non-stable with integrally small perturbations.

CONCLUSION

In the paper the problems of stability and non-stability of dynamical systems are considered. A new definition is given for stability - "stability with integrally small perturbations". For this, perturbation forces defined at the final interval of time are chosen from the class of generalized functions.

Necessary and satisfactory conditions are defined for which linear differential equations with constants coefficients are stable, non-stable and asymptotically stable with integrally small perturbations.

The method of constructing Lyapunov's function is given which satisfies stability, asymptotical stability and non-stability with integrally small perturbations for the systems of linear differential equations with constant coefficients.

Sufficient conditions have been obtained in which the trivial solution of non-linear static dynamic systems is stable or non-stable with integrally small perturbations.

The some of obtained main results were published in [11].

REFERENCES

[1] N.G. Chetaev, *Stability of Motion*, Nauka, 1990. In Russian.
[2] I.G. Malkin, *Theory of Stability of Motion*, Nauka, 1966. In Russian.
[3] N.N. Krasovsky, *Some problems in the theory of stability of motion*, Fizmatgiz, 1959. In Russian.
[4] J.L. Massera, *Ann. of Math.* **50**, 705-721 (1949).
[5] V.M. Matrosov, *Trudy of Kazan Aviation Institute* **45**, 63-76 (1959). In Russian.
[6] D.R. Merkin, *Introduction to the Theory of Stability*, Nauka, 1972. In Russian.
[7] A. Halanay and D. Wexler, *Qualitative theory of impulsive systems*, Mir, 1971. In Russian.

[8] A.M. Samoylenko and N.A. Perestyuk, *Differential equations with impulse action*, Visha School, 1987. In Russian.
[9] D.D. Bainov and P.S. Simeonov, *Systems with impulse effect. Stability*. *Theory and Applications,* Ellis Horwood Ltd, 1989.
[10] R.I. Gladilina and A.O. Ignatyev, Mathematical Notes, **76**, 44-51 (2004). In Russian.
[11] S.G. Shahinyan, *The Stability of Dynamic Systems with integrally small perturbations*, Publishing House "Scholars' Press", 2015.