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Mathematics

ON DIVERGENCE OF FOURIER–WALSH SERIES OF CONTINUOUS FUNCTION

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We prove that for any perfect set P of positive measure, for which 0 is a density point, one can construct a function $f(x)$ continuous on $[0,1)$ such that each measurable and bounded function, which coincides with $f(x)$ on the set P has diverging Fourier–Walsh series at 0.

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Introduction. Almost everywhere convergence and divergence problems of Fourier series in different classical orthonormal systems is one of the basic fields in Harmonic analysis. The following theorem was proved by Menshov [1]:

Theorem. For any perfect set $P \subset [-\pi,\pi]$ of positive measure, and for any density point x_0 of P one can define a continuous function $f(x)$ on $[-\pi,\pi]$, having the following property: any bounded measurable function $g(x)$, defined on $[-\pi,\pi]$ coinciding with $f(x)$ on P, has Fourier series diverging at x_0 with respect to the trigonometric system.

In this paper we prove the following theorem.

Theorem 1. For any perfect set $P \subset [0,1)$ of positive measure, for which 0 is a density point, one can define a continuous function $f(x)$ on $[0,1)$ with the following property: any bounded measurable function $g(x)$, defined on $[0,1)$ coinciding with $f(x)$ on P, has Fourier series diverging at 0 with respect to the Walsh system.

Definition of Walsh System. The Walsh system $\Phi = \{\phi(x)\}_{n=1}^{\infty}$ is defined as follows (see [2]):

$$\phi_0(x) = 1, \quad \phi_n(x) = \prod_{s=1}^{k} r_{m_s}(x) \ \text{ for } \ n = \sum_{s=1}^{k} 2^{m_s}, \ \ 0 \le m_1 < m_2 < \ldots < m_s,$$

where $\{r_k(x)\}_{k=0}^{\infty}$ is the Rademacher system:

$$r_0(x) = \begin{cases} 1, & x \in [0,1/2), \\ -1, & x \in [1/2,1); \end{cases}$$

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$$r_0(x+1) = r_0(x),\ r_k(x) = r_0(2^k x),\ k = 1,\ 2, \ldots$$

Note that the Walsh system is a basis in $L^p[0,1),\ 1 < p < \infty$.

Auxiliary Propositions. Let $D_n(x)$ be the Dirichlet kernel of the Walsh system and $S_n(x,f)$ be the partial sum of Fourier–Walsh series of a function $f(x)$, i.e.
$D_n(x) = \sum_{k=0}^{n-1} \phi_k(x),\ S_n(x,f) = \sum_{k=0}^{n-1} c_k\phi_k(x)$, where $c_k = \int_0^1 f(t)\phi_k(t)dt,\ k = 0,1,\ldots$

It is known [2] that

$$|D_n(x)| < \frac{1}{x},\ x \in (0,1),\ n = 1,2,\ldots, \tag{1}$$

$$S_n(x,f) = \int_0^1 f(t \oplus x) D_n(t)dt, \tag{2}$$

where $\oplus$ is the dyadic addition, and

$$\int_{2^{-k}}^1 |D_{n_k}(t)|dt \geq \frac{k}{4},\ k = 1,2,\ldots, \tag{3}$$

where

$$n_{2s} = \sum_{i=0}^{s-1} 2^{2i+1}, \quad n_{2s-1} = \sum_{i=0}^{s-1} 2^{2i}, \quad s = 1,2,\ldots \tag{3*}$$

Let $P \subset [0,1)$ be a perfect set of positive measure and $E = [0,1) \setminus P$. Let 0 be a density point of the set $P \subset [0,1)$, i.e.

$$\exists \lim_{h \to +0} \frac{|E \cap (-h,h)|}{2h} = 0.$$

We will also use the following lemma, which is a direct consequence of Lemma C from [1].

Lemma. There exists a positive function $\sigma(\alpha),\ \alpha \in (0,1)$ with $\lim_{\alpha \to +0} \sigma(\alpha) = 0$ such that

$$0 \leq \int\limits_{E \cap [\alpha_m, \alpha]} \frac{dt}{t} \leq \sigma(\alpha) \int_{\alpha_m}^{\alpha} \frac{dt}{t},\ m = 0,1,\ldots, \text{ where } \alpha_m = \frac{\alpha}{2^m}.$$

Proof of Main Result. We choose a sequence of natural numbers $\{k_m\}_{m=0}^{\infty}$ such that

$$k_0 = 1,\ k_m > m^2 k_{m-1},\ m = 1,2,\ldots, \tag{4}$$

$$\sigma\left(\frac{1}{2^{k_m}}\right) < \frac{1}{(m+1)^2},\ m = 1,2,\ldots \tag{5}$$

Denote

$$\Delta_m = \left[\frac{1}{2^{k_m}}, \frac{1}{2^{k_{m-1}}}\right) (k_{-1} = 0),\ \delta_m^i = \left[\frac{i}{2^{k_m}}, \frac{i+1}{2^{k_m}}\right),\ \gamma_m^{0,i} = \left[\frac{i}{2^{k_m}}, \frac{i}{2^{k_m}} + l_m\right),$$

$$\gamma_m^{1,i} = \left[\frac{i+1}{2^{k_m}} - l_m, \frac{i+1}{2^{k_m}}\right),\ i = 0,\ldots,2^{k_m}-1,\ m = 0,1,\ldots, \tag{6}$$

where $l_m = 1/(2^{2k_m+2})$.

From (3*) obviously follows, that for all $m \geq 0$ the function $D_{n_{k_m}}(x)$ is constant on each $\delta_m^i \subset \Delta_m$.

Then

$$[0,1) = \delta_m^0 \cup \Delta_m \cup \left[2^{-k_{m-1}}, 1\right), \ \Delta_m = \bigcup_{i=1}^{2^{k_m - k_{m-1}} - 1} \delta_m^i, \ m = 1, 2, \ldots \tag{7}$$

We define functions $f_0(x)$ and $f(x)$ as follows,

$$f_0(x) = \begin{cases} \dfrac{1}{m} \operatorname{sign} D_{n_{k_m}}(x), & \text{if } x \in \Delta_m, m = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

$$f(x) = \begin{cases} f_0\left(\frac{i}{2^{k_m}}\right), & \text{if } x \in \delta_m^i \setminus (\gamma_m^{0,i} \cup \gamma_m^{1,i}) \subset \Delta_m, \\ \dfrac{1}{l_m}\left(x - \frac{i}{2^{k_m}}\right) f_0\left(\frac{i}{2^{k_m}}\right), & \text{if } x \in \gamma_m^{0,i} \subset \Delta_m, \\ -\dfrac{1}{l_m}(x - \frac{i+1}{2^{k_m}}) f_0\left(\frac{i}{2^{k_m}}\right), & \text{if } x \in \gamma_m^{1,i} \subset \Delta_m, \\ 0, & \text{if } x = 0, \end{cases} \tag{9}$$

where $m = 0, 1, \ldots$

From (6)–(9) it is easy to notice that $f(x)$ is continuous on $[0,1)$.

Let $g(x)$ be an arbitrary measurable and bounded function defined on $[0,1)$ and coinciding with $f(x)$ on P. Then let m be a natural number. We put

$$I_m^{(1)} = \int_{\delta_m^0} g(t) D_{n_{k_m}}(t) dt, \ I_m^{(2)} = \int_{\Delta_m} g(t) D_{n_{k_m}}(t) dt, \ I_m^{(3)} = \int_{[2^{-k_{m-1}}, 1)} g(t) D_{n_{k_m}}(t) dt. \tag{10}$$

From (2), (7) and (10) we get

$$S_{n_{k_m}}(0, g) = \int_0^1 g(t) D_{n_{k_m}}(t) dt = I_m^{(1)} + I_m^{(2)} + I_m^{(3)}. \tag{11}$$

It follows from (6) and (10) that

$$|I_m^{(1)}| \leq C \int_{\delta_m^0} |D_{n_{k_m}}(t)| dt = C \frac{n_{k_m}}{2^{k_m}} \leq C, \quad C = \sup_{x \in [0,1)} g(x). \tag{12}$$

From (1) and (10) we obtain

$$|I_m^{(3)}| \leq C \int_{[2^{-k_{m-1}}, 1)} |D_{n_{k_m}}(t)| dt \leq C \int_{[2^{-k_{m-1}}, 1)} \frac{1}{t} dt = C k_{m-1} \ln 2. \tag{13}$$

Then

$$I_m^{(2)} = \int_{P \cap \Delta_m} g(t) D_{n_{k_m}}(t) dt \ + \int_{E \cap \Delta_m} g(t) D_{n_{k_m}}(t) dt.$$

Obviously

$$I_m^{(2)} = B_m^{(1)} + B_m^{(2)}, \tag{14}$$

where

$$B_m^{(1)} = \int_{\Delta_m} f(t) D_{n_{k_m}}(t) dt, \ B_m^{(2)} = \int_{E \cap \Delta_m} [g(t) - f(t)] D_{n_{k_m}}(t) dt. \tag{15}$$

From (1), (5), (15) and Lemma we conclude

$$|B_m^{(2)}| \leq 2C\sigma(2^{-k_{m-1}}) \int\limits_{\Delta_m} \frac{1}{t} dt \leq \frac{2C}{m^2}(k_m - k_{m-1}) \ln 2. \tag{16}$$

From (8) and (15) we have

$$B_m^{(1)} = \frac{1}{m} \int\limits_{\Delta_m} |D_{n_{k_m}}(t)| dt - \int\limits_{\Delta_m} [f_0(t) - f(t)] D_{n_{k_m}}(t) dt. \tag{17}$$

From (1), (3) and (6)–(9) we obtain

$$\int\limits_{\Delta_m} |f_0(t) - f(t)||D_{n_{k_m}}(t)| dt < 1, \int\limits_{\Delta_m} |D_{n_{k_m}}(t)| dt \geq \frac{k_m}{4} - k_{m-1} \ln 2. \tag{18}$$

From (11)–(14) and (16)–(18) we get

$$S_{n_{k_m}}(0,g) > \frac{1}{m}\left(\frac{k_m}{4} - k_{m-1} \ln 2\right) - \frac{2C}{m^2}(k_m - k_{m-1}) \ln 2 - Ck_{m-1} \ln 2 - C - 1. \tag{19}$$

From (4) and (19) it follows that

$$S_{n_{k_m}}(0,g) \to \infty \text{ when } m \to \infty,$$

which completes the proof of the Theorem.

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R E F E R E N C E S

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